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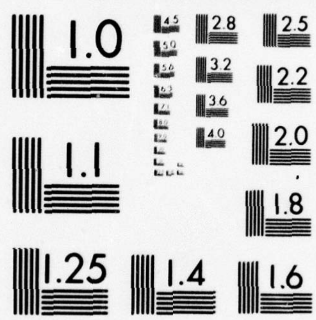

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This paper summarizes the results known to date for using sinc functions composed with other functions as bases for approximations in numerical analysis. Described in this paper are methods of interpolation and approximation of functions and their derivatives, quadrature, the approximate evaluation of transforms (Hilbert, Fourier, Laplace, Hankel and Mellin) and the approximate solution of differential and integral equations. The methods have many advantages over classical methods which use polynomials as bases. In addition, all of the methods converge at an optimal rate, if singularities on the boundary of approximation are ignored.

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Numerical Methods for Singularities via Sinc Functions*

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TABLE OF CONTENTS

Page

ACKNOWLEDGEMENTS. v

CHAPTERS

1. INTRODUCTION AND SUMMARY 1

2. PROPERTIES OF THE CARDINAL FUNCTION. 6

3. APPROXIMATIONS ON THE REAL LINE. 14

3.1. The Error of Approximation by $C(f,h)$. 16
3.2. The Error of Quadrature by the Trapezoidal Rule. . . . 18
3.3. Fourier Transforms on the Real Line. 19
3.4. Approximation of Derivatives 21
3.5. The Indefinite Integral. 24
3.6. Hilbert and Related Transforms 26
3.7. Singularities on the Interval of Approximation 29

4. FORMULAS OVER FINITE, SEMI-INFINITE INTERVALS AND CONTOURS . 33

4.1. Interpolation over Γ. 39
4.2. Quadrature over Γ 43
4.3. Approximation of Derivatives on Γ 45
4.4. Approximation of the Indefinite Integral on Γ 49
4.5. Singular Integrals on Γ 51

5. APPROXIMATION OF TRANSFORMS ON Γ. 66

6. APPROXIMATE SOLUTION OF DIFFERENTIAL EQUATIONS VIA THE SINC-GALERKIN METHOD 69

7. APPROXIMATE SOLUTION OF INTEGRAL EQUATIONS 83

8. COMPUTER IMPLEMENTATION AND PITFALLS 93

8.1. Computer Algorithms. 93
8.2. Pitfalls in Computation. 96

9. OPTIMALITY OF THE APPROXIMATIONS 99

REFERENCES. 105

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I would like to dedicate this paper to my good friends:

To E. L. Whitney (1921-1966), for inspiring in me my love for analytic functions;

To J. J. McNamee, for introducing me to the beautiful cardinal functions; and

To P. J. Davis, from whom I acquired my desire to ignore the singularity.

1. INTRODUCTION AND SUMMARY.

Most numerical approximation processes, such as interpolation, quadrature, finite difference approximation, finite element methods, and so on, are based on enact relationships that polynomials (and less frequently, trigonometric polynomials) satisfy. The simplicity and variety of these identities have made it possible to construct a vast number of different procedures. These procedures generally do very well in a region where the function to be approximated is analytic, and very poorly in a neighborhood of a singularity of the function.

This paper deals with the approximation of analytic functions f on an interval, or on a contour. These functions f may or may not have a singularity, i.e., a point at which f' does not exist, at the end-points of the intervals or contours. All of the approximations are derived via the use of exact relationships that the function $C(f,h)$, i.e., Whittaker's cardinal function, satisfies. Corresponding to a function f defined on the real line $\mathbb{R}$, the function $C(f,h)$ is defined by

$$C(f,h) = \sum_{k=-\infty}^{\infty} f(kh)S(k,h) \tag{1.1}$$

whenever this series converges, where $h > 0$ is the step-size, and where

$$S(k,h)(x) = \frac{\sin[\frac{\pi}{h}(x-kh)]}{\frac{\pi}{h}(x-kh)} \quad . \tag{1.2}$$

The numerical approximation procedures reported in this paper, and which are obtainable via the use of $S(k,h)\circ\phi$ as basis functions, where ϕ

denotes a suitable transformation of an interval onto $\mathbb{R}$, have roughly the same accuracy whether or not the function to be approximated has a singularity at an end-point of an interval. In the absence of singularities, this accuracy is usually not as good as that obtainable via polynomial methods, but if singularities are present, this accuracy is much better than that of polynomial methods.

The function $C(f,h)$ was discovered by E. T. Whittaker [55] who studied the mathematical properties of this function and who used it as a means of obtaining alternate expressions of entire functions. He called $C(f,h)$ "a function of royal blood in the family of entire functions, whose distinguished properties separate it from its bourgeois brethren". The study of this function was later continued and considerably extended by J. M. Whittaker [56,57]. The function $C(f,h)$ then played an important role in engineering applications in the transmission of information as a convenient approximation of f (Hartley [16], Nyquist [30], Shannon [37]). Engineers have since referred to $C(f,h)$ as the "band limited" or "sinc function" expansion of f .

The mathematical study of the accurate trapezoidal formula approximation of the integral of a function f over $\mathbb{R}$,

$$(1.3) \qquad \int_{\mathbb{R}} f(x)dx \cong h \sum_{k=-\infty}^{\infty} f(kh) ,$$

developed independently of the study of $C(f,h)$, although this approximation is identical to the integral of $C(f,h)$ over $\mathbb{R}$. It was Goodwin [14] who seemed to be the first to note the incredible accuracy of this formula for approximating the integral of certain functions that are analytic in a strip about the real axis. The incredible accuracy of $C(f,h)$ and of the trapezoidal rule as approximation tools in the family of functions that are analytic in a strip about the real line was later demonstrated in [25,26,27,54].

The application of the trapezoidal formula for approximating an integral over an interval other than $\mathbb{R}$ was investigated in [28,34,39], via the use of transformations, and it was shown in [45,48] that the most effective transformations are those that are a conformal map of the domain of analyticity of the integrand onto a strip about the real axis. Later [49], this transformation idea was used to construct interpolation and approximation formulas for other intervals as well as for carrying out the approximate solution of differential [53] and integral equations [18,19,32,36,46,47]. Most recently [52] all of these approximations were shown to have the optimal functional form $O(e^{-cn^{1/2}})$ for the rate of convergence of the error of an n-point approximation, whether or not the function being approximated has singularities at the end-points of the interval (or contour) of approximation.

Although the present paper is mainly a summary paper, some of the results in it, such as the results pertaining to the case where a function has a singularity on the interval of approximation, are new.

The function $C(f,h)$ is replete with beautiful properties and formulas. The known properties relevant to approximations are summarized in Sec. 2 of this paper. These properties are basic to the approximation procedures in later sections of the paper.

In Sec. 3 of the paper we define a space of functions that are analytic in a strip about the real line. In this space the identities of the previous sections are no longer exact, but highly accurate, as shown by the error bounds.

In Sec. 4 some of the approximations of Sec. 3 are extended to an arbitrary contour by use of conformal mapping [23,49]. Special attention is given to the important intervals $[0,1]$, $[-1,1]$, and $[0,\infty]$, and examples are given, of approximation rules for these intervals.

Section 5, 6, 7 and 8 deal with applications of the previously constructed approximation procedures.

In Sec. 5 we consider the approximations of important transforms over the interval $(0,\infty)$: the Laplace, the semi-infinite Fourier, the Mellin and the Hankel transforms [22].

Sec. 6 deals with approximate procedures for solving ordinary and partial differential equation boundary value problems [53]. It is here that the approximation procedures of this paper are particularly powerful, especially in the cases where the singular behavior of the solution on the boundary is not known explicitly. The functions $S(k,h)\circ\phi$ (or product of these for the case of more than one dimension) are the basis functions. These make it possible to explicitly write down highly accurate expressions of the inner products in the Galerkin scheme which reduces the differential equation problem to an algebraic problem, even for the case of nonlinear differential equations. Examples of the approximate solution of "model" problems illustrate the application of the method.

In Sec. 7 of the paper we apply some of the approximations in earlier parts of the paper to the approximate solution of (singular) integral equations. Examples are given, illustrating the approximation procedures. Here, too, the functions $S(k,h)\circ\phi$ are very well suited for easily obtaining accurate approximate solutions.

In Sec. 8 we summarize the main ideas used for the implementation of the methods on a computer, and we list already existing computer algorithms. In addition we caution the user against possible computational pitfalls resulting from inaccurate (or improper) numerical evaluation of a function in the neighborhood of a singularity.

Sec. 9 discusses rates of convergence of the methods of this paper, and compares these with rates of polynomial methods. It shows, moreover, that the $O(e^{-cn^{1/2}})$ rate of convergence of the methods of this paper cannot be improved by any other methods of approximation.

2. PROPERTIES OF THE CARDINAL FUNCTION.

Just as polynomials satisfy certain exact relationships in the space of polynomials of the same degree, the same is true of the cardinal function in a certain space of entire functions.

DEFINITION 2.1.: Let $h > 0$, and let $B(h)$ denote the family of functions f that are analytic in the entire complex plane $\mathbb{C}$, such that

$$|f(z)| \leq Ce^{\pi|z|/h} \quad , \tag{2.1}$$

and such that $f \in L^2(\mathbb{R})$, where $\mathbb{R} = (-\infty,\infty)$. Let k be an integer, and let us set

$$S(k,h)(z) = \frac{\sin[\frac{\pi}{h}(z-kh)]}{\frac{\pi}{h}(z-kh)} \quad . \tag{2.2}$$

If f is defined on $\mathbb{R}$, the Whittaker cardinal function for f with step-size h is defined by

$$C(f,h)(z) \equiv \sum_{k=-\infty}^{\infty} f(kh)S(k,h)(z) \quad . \tag{2.3}$$

Set

$$\delta_{jk}^{(n)} = S^{(n)}(j,1)(k) = \left(\frac{d}{dx}\right)^n S(j,1)(x)\Big|_{x=k} \tag{2.4}$$

In particular

$$\delta_{jk}^{(0)} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j\neq k \end{cases}$$

(2.4a)
$$\delta_{jk}^{(1)} = \begin{cases} 0 & \text{if } j=k \\ \dfrac{(-1)^{k-j}}{k-j} & \text{if } j\neq k \end{cases}$$

$$\delta_{jk}^{(2)} = \begin{cases} -\pi^2/3 & \text{if } j=k \\ \dfrac{-2(-1)^{k-j}}{(k-j)^2} & \text{if } j\neq k \end{cases} .$$

The following theorem summarizes the known explicit relations involving $C(f,h)$ and $S(k,h)$, in the case when $f \in B(h)$.

<u>THEOREM 2.1:</u> Let $f \in B(h)$. Then:

(a)

(2.5)
$$f(z) = C(f,h)(z) \quad \text{for all} \quad z \in \mathbb{C} ;$$

(b)

(2.6)
$$\int_R f(z)dz = h \sum_{k=-\infty}^{\infty} f(k,h) \quad ;$$

(c)

(2.7)
$$\int_R |f(x)|^2 dx = h \sum_{k=-\infty}^{\infty} |f(kh)|^2 \quad ,$$

and the set $\{h^{-1/2}S(k,h)\}_{k=-\infty}^{\infty}$ is therefore a complete orthonormal sequence in $B(h)$;

(d) There exists a unique function $g \in L^2(-\pi/h,\pi/h)$, such that

(2.8)
$$f(z) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-izt} g(t)dt \quad ;$$

(e) The function g in (d) is given by

(2.9)
$$\int_R e^{ixt} f(t)dt = \begin{cases} h \sum_{k=-\infty}^{\infty} f(kh)e^{ikhx} \\ \quad \text{if} \quad -\frac{\pi}{h} < x < \frac{\pi}{h} \\ 0 \quad \text{if} \quad x > \pi/h \quad \text{or if} \quad x < -\frac{\pi}{h} \quad ; \end{cases}$$

(f)

(2.10)
$$f(z) = \frac{1}{h}\int_R f(t) \frac{\sin \frac{\pi}{h}(z-t)}{\frac{\pi}{h}(z-t)} \, dt \quad ;$$

(g)

(2.11)
$$f' \in B(h) \quad ;$$

(h)

(2.12)
$$f^{(n)}(kh) = h^{-n} \sum_{j=-\infty}^{\infty} \delta_{jk}^{(n)} f(jh)$$

where $\delta_{jk}^{(n)}$ is defined in (2.4), and therefore, by (2.12),

(2.13)
$$f^{(n)}(x) = h^{-n} \sum_{k=-\infty}^{\infty} [\sum_{j=-\infty}^{\infty} \delta_{jk}^{(n)} f(jh)] \, S(k,h)(x)$$

(i) Let g be defined as in (2.8). Then

(2.14)
$$\int_0^x f(t)dt = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} g(\xi) \frac{1-e^{-i\xi x}}{i\xi} \, d\xi \quad ,$$

and in particular,

(2.15)
$$\int_0^x S(k,h)(t)dt = h[\sigma_k + \frac{1}{\pi}\int_0^{\pi} \frac{\sin[(\frac{x}{h} - k)\xi]}{\xi} d\xi \,]$$

where (see Table 2.1)

(2.16)
$$\sigma_k = \frac{1}{\pi}\int_0^{\pi} \frac{\sin k\xi}{\xi} \, d\xi \quad .$$

Moreover, if $\int_{\mathbb{R}} f(t)dt = 0$, and if $\int_{-\infty}^{x} f(t)dt$ is in $B(h)$, then

(2.17) $$\int_{-\infty}^{x} f(t)dt = h \sum_{k=-\infty}^{\infty} [\sum_{j=-\infty}^{\infty} \sigma_{k-j} f(jh)]S(k,h)(x) \quad ,$$

(j) Let Pf and Hf be defined by

(2.18) $$(Pf)(x) = \lim_{y\to 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-x-iy} dt \; .$$

(2.19) $$(Hf)(x) = \frac{P.V.}{\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-x} dt \; .$$

Then

(2.20) $$(Pf)(x) = \frac{1}{2i} \sum_{k=-\infty}^{\infty} f(kh) \left[\frac{e^{i\pi(x-kh)/h}-1}{\pi(x-kh)/h}\right]$$

and, since $Pf \in B(h)$,

(2.21) $$(Pf)(x) = \sum_{k=-\infty}^{\infty} \{\tfrac{1}{2}f(kh) + \frac{h}{2\pi i} \sum_{\substack{j=-\infty \\ j\neq k}}^{\infty} f(jh)[\frac{(-1)^{k-j}-1}{k-j}]\} \; S(k,h)(x) \; ;$$

Similarly

(2.22) $$(Hf)(x) = i \sum_{k=-\infty}^{\infty} f(kh) \frac{\pi}{2h} (x-kh) S^2(0,1) \circ [\frac{\pi}{2h}(x-kh)] \quad ,$$

and since $Hf \in B(h)$,

(2.23) $$(Hf)(x) = \frac{i}{\pi} \sum_{k=-\infty}^{\infty} \{ \sum_{\substack{j=-\infty \\ j\neq k}}^{\infty} f(jh) [\frac{1-(-1)^{k-j}}{k-j}]\} S(k,h)(x) \quad ;$$

(k) Let $a > 0$, and let $\alpha \geq -\frac{1}{2}$. Then

$$\int_{\mathbf{R}} \frac{f(t)dt}{[(x-t)^2+a^2]^{\alpha+1/2}}$$

(2.24)

$$= \frac{(2a)^{-\alpha}}{\pi^{1/2}\Gamma(\alpha+\frac{1}{2})} \int_{-\pi/h}^{\pi/h} e^{-ixt} g(t)|t|^{\alpha} K_{\alpha}(a|t|)\, dt$$

where g is defined in terms of f, by (2.8), where K_{α} denotes the Bessel function,

(2.25) $$K_{\alpha}(x) = \frac{1}{2}\pi[\sin(\pi\alpha)]^{-1}[e^{i\alpha\pi/2}J_{-\alpha}(ix) - e^{-i\pi\alpha/2}J_{\alpha}(ix)]$$

and where

(2.26) $$J_{\alpha}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+\alpha}}{m!\ \Gamma(m+\alpha+1)} .$$

In particular

(2.27) $$\tau_k(a,\alpha,h;x) \equiv \int_{\mathbf{R}} \frac{S(k,h)(t)dt}{[(x-t)^2+a^2]^{\alpha+1/2}}$$

$$= \frac{2\ (2ah)^{-\alpha}}{\pi^{1/2}\ \Gamma(\alpha+1/2)} \int_0^{\pi} \cos[(\frac{x}{h}-k)t]t^{\alpha}K_{\alpha}(aht)dt$$

and

(2.28) $$\tau_k(a,\alpha,h;\ell h) = \tau_{k-\ell}(a,\alpha,h)$$

where

(2.29) $$\tau_k(a,\alpha,h) = \frac{2\ (2ah)^{-\alpha}}{\pi^{1/2}\ \Gamma(\alpha+1/2)} \int_0^{\pi} \cos(kt)t^{\alpha}K_{\alpha}(aht)dt .$$

Hence

$$\int_{\mathbb{R}} \frac{f(t)dt}{[(x-t)^2+a^2]^{\alpha+1/2}} = \sum_{k=-\infty}^{\infty} f(kh)\tau_k(a,\alpha,h;x) \tag{2.30}$$

and if this function is in $B(h)$, then

$$\int_{\mathbb{R}} \frac{f(t)dt}{[(x-t)^2+a^2]^{\alpha+1/2}} = \sum_{k=-\infty}^{\infty} \{ \sum_{j=-\infty}^{\infty} f(jh)\tau_{k-j}(a,\alpha,h)\} S(k,h)(x) \quad . \tag{2.31}$$

In the cases when $\alpha+1/2 \geq 0$ is an integer, the functions $\tau_k(a,\alpha,h;x)$ can be evaluated explicitly. In particular, if $\alpha=1/2$, then (2.30) becomes

$$\begin{aligned} &\frac{y}{\pi}\int_{\mathbb{R}} \frac{f(t)dt}{[(x-t)^2+y^2]} \\ &= \frac{h}{\pi}\sum_{k=-\infty}^{\infty} f(kh)\left\{\frac{y(1-e^{-\pi y/h})\cos[\pi(x-kh)/h] + (x-kh)e^{-\pi y/h}\sin[\pi(x-kh)/h]}{(x-kh)^2+y^2}\right\} ; \end{aligned} \tag{2.32}$$

(ℓ)

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{R}} f(t)\log[(x-t)^2+y^2]\, dt \\ &= \frac{h}{2\pi}\sum_{k=-\infty}^{\infty} f(kh)\left\{-\gamma-\log \pi/h + [1-\cos\{\pi(x-kh)/h\}]\log\left[\frac{(x-kh)^2+y^2}{(x-kh)^2}\right]\right. \\ &\qquad \left. + \int_0^{\pi} \frac{1-(2-e^{-yt/h})\cos\{(x-kh)t/h\}}{t}\, dt\right\} \end{aligned} \tag{2.33}$$

where γ denotes Euler's constant;

(m) Let $0 < \alpha < 1$, and let g be defined as in (2.8). Then

$$\int_{\mathbb{R}} |x-t|^{\alpha-1}f(t)dt = \frac{\Gamma(\alpha)\cos(\pi\alpha/2)}{2\pi}\int_{-\pi/h}^{\pi/h} |t|^{-\alpha}g(t)e^{-ixt}dt \quad , \tag{2.34}$$

and in particular

$$(2.35)\qquad \int_{\mathbb{R}} |x-t|^{\alpha-1} S(k,h)(t)dt = \frac{h^{\alpha}\Gamma(\alpha)\cos(\pi\alpha/2)}{\pi} \int_0^{\pi} t^{-\alpha}\cos[(x-kh)t/h]\, dt \quad ,$$

so that

$$(2.36)\qquad \int_{\mathbb{R}} |x-t|^{\alpha-1} f(t)dt = \frac{\Gamma(\alpha)\cos(\pi\alpha/2)}{2\pi}\, h^{\alpha} \cdot$$

$$\cdot \sum_{k=-\infty}^{\infty} f(kh) \int_0^{\pi} t^{-\alpha}\cos[(x-kh)t/h]\, dt \quad .$$

These results follow from the results in (k) above;

(n) Let g be defined as in (2.8). Then

$$(2.37)\qquad \int_{\mathbb{R}} \log|x-t|\; f(t)dt$$

$$= \frac{1}{4}\int_{-\pi/h}^{\pi/h} \left[\frac{g(0)-e^{-ixt}g(t)}{|t|}\right] dt - \frac{1}{2}\,[\gamma+\log(\pi/h)]\; g(0) \quad ,$$

and in particular

$$(2.38)\qquad \int_{\mathbb{R}} \log|x-t|\; S(k,h)(t)dt$$

$$= -\frac{1}{2}h \left\{ \gamma+\log(\pi/h) - \int_0^{\pi} \frac{1-\cos[(x-kh)t/h]}{t}\, dt \right\} \quad ,$$

so that

$$(2.39)\qquad \int_{\mathbb{R}} \log|x-t|\; f(t)dt$$

$$= -\frac{1}{2}h \sum_{k=-\infty}^{\infty} f(kh) \left\{ \gamma+\log(\pi/h) - \int_0^{\pi} \frac{1-\cos[(x-kh)t/h]}{t}\, dt \right\} \quad .$$

This result is obtained from (2.33), by letting $y \to 0$ there.

TABLE 2.1

Integrals of the Sinc Function from 0 to n : $\sigma_n = \int_0^n \frac{\sin \pi x}{\pi x}\, dx$

n	integral	n	integral
1	0.539489872236083636d+00	51	0.501986535165779514d+00
2	0.451411666790140314d+00	52	0.498051661656333795d+00
3	0.533093237618271984d+00	53	0.501911582593445087d+00
4	0.474969669883655078d+00	54	0.498123812121572928d+00
5	0.520107164191308518d+00	55	0.501842079980753156d+00
6	0.483205217497747133d+00	56	0.498190810017933564d+00
7	0.514415997123305252d+00	57	0.501777453798825155d+00
8	0.487374225057819973d+00	58	0.498253188234334582d+00
9	0.511230152636997458d+00	59	0.501717208261115709d+00
10	0.489888171153878660d+00	60	0.498311408629275600d+00
11	0.509195742008216617d+00	61	0.501660912583309053d+00
12	0.491568351668600880d+00	62	0.498365873483410454d+00
13	0.507784657812566074d+00	63	0.501608190668941573d+00
14	0.492770209374803135d+00	64	0.498416934805585154d+00
15	0.506748694472011579d+00	65	0.501558712698511542d+00
16	0.493672415178220958d+00	66	0.498464901947476010d+00
17	0.505955907917176832d+00	67	0.501512188224493392d+00
18	0.494374552833662521d+00	68	0.498510047874920881d+00
19	0.505329710440158862d+00	69	0.501468360466826563d+00
20	0.494936499570695472d+00	70	0.498552614364505956d+00
21	0.504822607304171877d+00	71	0.501427001572250365d+00
22	0.495396415084857158d+00	72	0.498592816334313617d+00
23	0.504403585198050434d+00	73	0.501387908652699940d+00
24	0.495779766136643166d+00	74	0.498630845472585081d+00
25	0.504051535843944646d+00	75	0.501350900457384930d+00
26	0.496104197488515347d+00	76	0.498666873293589259d+00
27	0.503751595031909162d+00	77	0.501315814563370501d+00
28	0.496382320165352064d+00	78	0.498701053723476824d+00
29	0.503492993277940418d+00	79	0.501282504992800478d+00
30	0.496623386631180065d+00	80	0.498733525298349206d+00
31	0.503267736947771822d+00	81	0.501250840183041950d+00
32	0.496834338855222759d+00	82	0.498764413040732243d+00
33	0.503069768202187467d+00	83	0.501220701250237743d+00
34	0.497020487027667043d+00	84	0.498793830068040258d+00
35	0.502894412555701326d+00	85	0.501191980497952594d+00
36	0.497185962335964750d+00	86	0.498821878976649611d+00
37	0.502738005382438294d+00	87	0.501164580131481855d+00
38	0.497334026926906631d+00	88	0.498848653037272067d+00
39	0.502597633215927650d+00	89	0.501138411145478520d+00
40	0.497467290977560678d+00	90	0.498874237230974993d+00
41	0.502470950690821481d+00	91	0.501113392358239876d+00
42	0.497587867804393108d+00	92	0.498898709150092725d+00
43	0.502356048524901335d+00	93	0.501089449570580461d+00
44	0.497697486705937335d+00	94	0.498922139784147706d+00
45	0.502251356677318093d+00	95	0.501066514830934817d+00
46	0.497797576391022357d+00	96	0.498944594207547635d+00
47	0.502155572214122969d+00	97	0.501044525791360742d+00
48	0.497889327564668397d+00	98	0.498966132183088041d+00
49	0.502067604827451950d+00	99	0.501023425141590715d+00
50	0.497973740503081827d+00	100	0.498986808693045503d+00

3. APPROXIMATIONS OVER THE REAL LINE.

Whereas the relationships of the previous section are exact, each of the formulas (2.4), (2.6), and (2.9) provides a method of approximation for the case when f does not belong to the class $B(h)$. We thus introduce another class of functions defined on $\mathbb{R} = (-\infty,\infty)$, for which the approximations referred to above are extremely accurate. At the outset we investigate the error of the approximations for the case when the complete set of points $\{kh\}_{k=-\infty}^{\infty}$ are used in the approximations.* We then also investigate the error of approximation when only the finite set of points $\{kh\}_{k=-N}^{N}$ is used, and h is chosen judiciously.

<u>DEFINITION 3.1.:</u> Let $d > 0$, and let $\mathcal{D}_d$ denote the domain

$$\mathcal{D}_d = \{z \in \mathbb{C}: \ |\mathrm{Im}\, z| < d\} . \tag{3.1}$$

Let $p \geq 1$, and let $B_p(\mathcal{D}_d)$ denote the family of all functions f that are analytic in $\mathcal{D}_d$, such that

$$\int_{-d}^{d} |f(x+iy)|\,dy \to 0 \quad \text{as} \quad x \to \pm\infty \ ; \tag{3.2}$$

and such that $N_p(f,\mathcal{D}_d) < \infty$, where

$$N_p(f,\mathcal{D}_d) = \lim_{y\to d^-}\left\{\left(\int_{\mathbb{R}} |f(x+iy)|^p dx\right)^{1/p} + \left(\int_{\mathbb{R}} |f(x-iy)|^p dx\right)^{1/p}\right\} . \tag{3.3}$$

---

*An exception occurs, of course, for the case of Eq. (2.10).

If $p=1$, we shall simply write $B(\mathcal{D}_d)$ and $N(f,\mathcal{D}_d)$ instead of $B_1(\mathcal{D}_d)$ and $N_1(f,\mathcal{D}_d)$ respectively.

Let us set

$$\text{(3.4)}\quad \begin{cases} S(k,h)(x) = \dfrac{\sin[\frac{\pi}{h}(x-kh)]}{\frac{\pi}{h}(x-kh)} \\[2ex] C(f,h) = \displaystyle\sum_{k=-\infty}^{\infty} f(kh)S(k,h) \\[2ex] C_N(f,h) = \displaystyle\sum_{k=-N}^{N} f(kh)S(k,h) \\[2ex] E(f,h) = f - C(f,h) \\[2ex] E_N(f,h) = f - C_N(f,h) \end{cases}$$

The most effective application of the formulas of this section occur for the case when $f \in B(d)$, and when

$$\text{(3.5)}\qquad |f(x)| < Ce^{-\alpha|x|}$$

for all $x \in \mathbb{R}$, where C and α are positive constants.

FIGURE 3.1 The Region $\mathcal{D}_d$ of Eg. (3.1).

3.1 Error of Approximation by $C(f,h)$.

The error of approximation of a function f in $B_p(\mathcal{D}_d)$ by $C(f,h)$ may be expressed explicitly as an integral [27].

THEOREM 3.1: Let $f \in B_p(\mathcal{D}_d)$. Then for all $x \in \mathbb{R}$,

$$E(f,h)(x) = \frac{\sin(\frac{\pi x}{h})}{2\pi i} \int_{\mathbb{R}} \left\{ \frac{f(t-id^-)}{(t-x-id)\sin[\frac{\pi}{h}(t-id)]} \right. \tag{3.6}$$

$$\left. - \frac{f(t+id^-)}{(t-x+id)\sin[\frac{\pi}{h}(t+id)]} \right\} dt$$

This result forms the starting point for obtaining the error of many different types of approximations. In particular the following theorem was proved in [46].

THEOREM 3.2: (a) If $f \in B(\mathcal{D}_d)$, then*

$$2\pi d \, \|E(f,h)\|_\infty \; , \; 2(\pi d)^{1/2} \|E(f,h)\|_2 \leq \frac{N(f,\mathcal{D}_d)}{\sinh(\pi d/h)} \; . \tag{3.7}$$

(b) If $f \in B_2(\mathcal{D}_d)$, then

$$\|E(f,h)\|_2 \; , \; 2(\pi d)^{1/2} \, \|E(f)\|_\infty \leq \frac{N_2(f,\mathcal{D}_d)}{\sinh(\frac{\pi d}{h})} \; . \tag{3.8}$$

(c) If $f \in B_p(\mathcal{D}_d)$, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

*Actually, the bound on $\|E(f,h)\|_2$ is new. It is obtained via a direct application of the inequality $\|\phi\|_2 \leq \|f\|_2 \|g\|_1$, where $\phi(x) = \int_{\mathbb{R}} f(x-t)g(t)dt$, and where $f \in L^2(\mathbb{R})$, $g \in L^1(\mathbb{R})$, to (3.6).

then

$$(3.9) \qquad \|E(f,h)\|_\infty \leq \frac{1}{\pi(2d)^{1/p}} \left[\frac{\Gamma(\frac{q-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{q}{2})} \right]^{1/q} \frac{N_p(f,\mathcal{D}_d)}{\sinh(\frac{d}{h})} \quad .$$

We remark that once we have bounds on $\|E(f,h)\|_2$ and $\|E(f,h)\|_\infty$, we can get the bound on $\|E(f,h)\|_s$, for any s between 2 and ∞, via the inequality

$$(3.10) \qquad \|E(f,h)\|_s \leq \|E(f,h)\|_2^{2/s} \, \|E(f,h)\|_\infty^{1-2/s} \quad .$$

<u>COROLLARY 3.3 [49]:</u> Let the condition (a), (b) or (c) of Theorem 3.2 be satisfied and let f satisfy (3.5). Then by choosing $h = [\pi d/(\alpha N)]^{1/2}$

$$(3.11) \qquad \|E_N(f,h)\|_s \leq C_1 N^{1/2} e^{-(\pi d \alpha N)^{1/2}}$$

where C_1 depends only on f, d and α, and for appropriate* s, as in Theorem 3.2.

---

* If bounds on both $\|E(f,h)\|_2$ and $\|E(f,h)\|_\infty$ are given in Theorem 3.2, then (3.11) holds for all $s \in [2,\infty]$.

3.2 Error of Quadrature by the Trapezoidal Rule.

Let $f \in B(\mathcal{D}_d)$, and let us set

$$\begin{aligned} \eta_N(f,h) &= \int_R E_N(f,h)(x)\,dx \\ &= \int_R f(x)\,dx - h \sum_{k=-N}^{N} f(kh) \quad , \end{aligned} \tag{3.12}$$

and

$$\eta(f,h) = \lim_{N\to\infty} \eta_N(f,h) \quad . \tag{3.13}$$

The error $\eta(f,h)$ may be obtained by integrating (3.6) using residues. This yields [22,26,27,45] the following theorem.

<u>THEOREM 3.4:</u> Let $f \in B(\mathcal{D}_d)$, and let $\eta(f,h)$ be defined in (3.14).

(a) Then

$$\eta(f,h) = \frac{1}{2i} \int_{\mathbb{R}} \left\{ \frac{f(t+id^-)e^{-(d-it)\pi/h}}{\sin[\frac{\pi}{h}(t+id)]} - \frac{f(t-id^-)e^{-(d+it)\pi/h}}{\sin[\frac{\pi}{h}(t-id)]} \right\} dt \quad . \tag{3.14}$$

Moreover,

$$|\eta(f,h)| \leq \frac{1}{2} \frac{e^{-\pi d/h}}{\sinh(\pi d/h)} N(f,\mathcal{D}_d) \quad . \tag{3.15}$$

If in addition f satisfies (3.5), then by taking $h = [2\pi d/(\alpha N)]^{1/2}$

$$|\eta_N(f,h)| \leq C_1 e^{-(2\pi d\alpha N)^{1/2}} \quad , \tag{3.16}$$

where C_1 depends only on f, d and α .

3.3 Fourier Transforms.

In this section we shall give a bound on the error of the approximation used in the Fast Fourier Transform method [49].

Let $E(f,h)$ and $E_N(f,h)$ be defined as in (3.4), let $x \in \mathbb{R}$, and let us set

$$\delta_N(f,h)(x) = \int_{\mathbb{R}} e^{ixt} E_N(f,h)(t)dt$$

$$(3.17) \qquad = \begin{cases} \int_{\mathbb{R}} f(t)e^{ixt}dt - h\sum_{j=-N}^{N} f(jh)e^{ijhx}, & |x| < \frac{\pi}{h} \\ \int_{\mathbb{R}} f(t)e^{ixt}dt, & |x| > \frac{\pi}{h}, \end{cases}$$

and

$$(3.18) \qquad \delta(f,h) = \lim_{N\to\infty} \delta_N(f,h) \quad ,$$

Eq. (3.17) tells us that the sum $h\sum_{j=-N}^{N} f(jh)e^{ijhx}$ is not to be used to approximate $\int_{\mathbb{R}} e^{ixt}f(t)dt$ if $|x| > \pi/h$. By replacing $f(t)$ by $f(t)e^{ixt}$ in (3.15), we get

<u>THEOREM 3.5:</u> Let $f \in B(\mathcal{D}_d)$, let $x \in \mathbb{R}$, $|x| < \pi/h$, and let $\delta(f,h)$ be defined as in (3.19). Then

$$(3.19) \qquad \delta(f,h)(x) = \frac{1}{2i}\int_{\mathbb{R}} \left\{ \frac{f(u+id^-)e^{-(d-iu)(\frac{\pi}{h}+x)}}{\sin[\frac{\pi}{h}(u+id)]} - \frac{f(u-id^-)e^{-(d+iu)(\frac{\pi}{h}-x)}}{\sin[\frac{\pi}{h}(u-id)]} \right\} du \quad .$$

and so

$$|\delta(f,h)(x)| \leqq \frac{1}{2} \frac{N(f,\mathcal{D}_d)}{\sinh(\frac{\pi d}{h})} e^{-d(\frac{\pi}{h} - |x|)} \tag{3.20}$$

<u>COROLLARY 3.6 [49]:</u> Let the conditions of Theorem 3.5 be satisfied, let f satisfy (3.5) on $\mathbb{R}$, and let $h = (\pi d/\alpha N)^{1/2}$. Then

$$|\delta_N(f,h)(x)| \leqq C_1 e^{-(\pi d\alpha N)^{1/2}}, \quad |x| \leqq \pi/h \tag{3.21}$$

where C_1 is a constant depending only on f, d and α.

3.4 Approximation of Derivatives.

In applications it is often desirable to approximate both a function as well as some of its derivatives. These derivative approximations are readily obtainable by differentiating $C_N(f,h)$. Bounds on the error of approximation of $f^{(n)}$ on $\mathbb{R}$ by $C_N(f,h)^{(n)}$ are readily obtained by bounding the integral [25]

$$E(f,h)^{(n)}(x)$$

$$(3.22)\quad = \frac{n!}{2\pi i}\int_{\mathbb{R}} \left\{ \left| \sum_{j=0}^{n} \frac{\sin^{(n-j)}(\frac{\pi x}{h})(\pi/h)^{n-j}}{(n-j)!(t-x-id)^{j+1}} \right| \frac{f(t-id^-)}{\sin[\frac{\pi}{h}(t-id)]} - \left| \sum_{j=0}^{n} \frac{\sin^{(n-j)}(\frac{\pi x}{h})(\pi/h)^{n-j}}{(n-j)!(t-x+id)^{j+1}} \right| \frac{f(t+id^-)}{\sin[\frac{\pi}{h}(t+id)]} \right\} dt$$

which follows from Eq. (3.6). The details of bounding $E(f,h)^{(n)}$ and $E_N(f,h)^{(n)}$ are carried out in [25]. We state some of these results.

<u>THEOREM 3.7:</u> Let $n \geq 0$ be an integer. (a) Let $f \in B(\mathcal{D}_d)$, and let $\pi d/h > 1$. Then*

$$(3.23)\quad \|E(f,h)^{(n)}\|_2 \leq \frac{n!\ e^{1/2}}{2(\pi d)^{1/2}} \frac{N(f,\mathcal{D}_d)}{(1-\frac{h^2}{\pi^2 d^2})^{1/4}} \frac{(\pi/h)^n}{\sinh(\pi d/h)}$$

and

$$(3.24)\quad \|E(f,h)^{(n)}\|_\infty \leq \frac{n!\ e\ N(f,\mathcal{D}_d)}{2\pi d} \frac{(\pi/h)^n}{\sinh(\pi d/h)} ;$$

---

*See the footnote on page 16 re. the $\|E(f,h)^{(n)}\|_2$ - bound which is new.

(b) Let $f \in B_2(\mathcal{D}_d)$, and let $\pi d/h > 1$. Then

$$\|E(f,h)^{(n)}\|_2 \leqq \frac{n!\ e\ N_2(f,\mathcal{D}_d)}{2\pi} \quad \frac{(\pi/h)^n}{\sinh(\pi d/h)} \tag{3.25}$$

and

$$\|E(f,h)^{(n)}\|_\infty \leqq \frac{n!\ e^{1/2}\ N_2(f,\mathcal{D}_d)}{2(\pi d)^{1/2}(1-\frac{h^2}{a^2d^2})^{1/4}} \quad \frac{(\pi/h)^n}{\sinh(\pi d/h)} \tag{3.26}$$

<u>COROLLARY 3.8 [25]:</u> Let f satisfy the condition (a) or (b) of Theorem 3.7, and let f satisfy (3.5). Then by choosing $h = [\pi d/(\alpha N)]^{1/2}$

$$\begin{aligned} \|E_N(f,h)^{(n)}\|_s &= \|f^{(n)} - C_N(f,h)^{(n)}\|_s \\ &\leqq C_1 N^{\frac{n+1}{2}} e^{-(\pi d\alpha N)^{1/2}} \end{aligned} \tag{3.27}$$

for any s in $[2,\infty]$, where C_1 depends only on f, d, α and n . In particular, with $\delta_{jk}^{(n)}$ defined as in (2.4),

$$|f^{(n)}(kh) - h^{-n} \sum_{j=-N}^{N} f(jh)\delta_{jk}^{(n)}| \leqq C_1 N^{\frac{n+1}{2}} e^{-(\pi d\alpha N)^{1/2}} \tag{3.28}$$

<u>COROLLARY 3.9*:</u> Let the condition of Corollary 3.8 be satisfied, and on $\mathbb{R}$, let

$$|f^{(n)}(x)| \leqq C_2 e^{-\alpha|x|} \quad . \tag{3.29}$$

Then there exists a constant C_3 depending only on f, d, α and n , such that

*This result is believed to be new.

$$(3.30)\qquad \left|f^{(n)}(x) - h\sum_{k=-N}^{N}\left(\sum_{j=-N}^{N} f(jh)\delta_{jk}^{(n)}\right)S(k,h)(x)\right| \leqq C_3 N^{\frac{n+3}{2}} e^{-(\pi d\alpha N)^{1/2}}$$

The approximations (3.28) and (3.30) are useful in the solution of differential equations.

3.5 The Indefinite Integral.

Let us present an approximation of

$$I(x) = \int_{-\infty}^{x} f(t)dt \tag{3.31}$$

in terms of the values $f(kh)$ of f [21].

<u>THEOREM 3.10:</u> Let $f \in B(\mathcal{D}_d)$, and let $g \in B(\mathcal{D}_d)$ where

$$g(x) = I(x) - \frac{e^{\beta x}}{e^{\beta x}+e^{-\beta x}} I(\infty) \tag{3.32}$$

and where

$$0 < \beta < \frac{\pi}{2d} \quad . \tag{3.33}$$

On $\mathbb{R}$, let

$$|f(x)| \leq C e^{-\alpha'|x|} \tag{3.34}$$

and let

$$\alpha = \min(\alpha', 2\beta) \ . \tag{3.35}$$

Then for $h = [\pi d/(\alpha N)]^{1/2}$,

$$\begin{aligned} &\Bigg| \int_{-\infty}^{x} f(t)dt - \frac{e^{\beta x}}{e^{\beta x}+e^{-\beta x}} \int_{\mathbb{R}} f(t)dt \\ &- h \sum_{k=-N}^{N} \left\{ \sum_{j=-N}^{N} \sigma_{k-j} [f(jh) - \frac{2\beta}{(e^{\beta jh}+e^{-\beta jh})^2} \int_{\mathbb{R}} f(t)dt] \right\} S(k,h)(x) \Bigg| \\ &\leq C_1 N e^{-(\pi d \alpha N)^{1/2}} \end{aligned} \tag{3.36}$$

where σ_k is defined in (2.16), and where C_1 is a constant depending only on f, d and α.

In applications it usually suffices to take $\beta = 1/2$ or 1, changing d instead.

3.6 The Hilbert and Related Transforms.

Given $f \in B_p(\mathcal{D}_d)$, $p \geq 1$, each of the integrals: the projection integral

$$(3.37) \qquad (\mathcal{P}f)(x) = \lim_{y\to 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{f-x-iy}\, dt$$

and the Hilbert transform integral

$$(3.38) \qquad (\mathcal{H}f)(x) = \frac{\text{P.V.}}{\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-x}\, dt$$

clearly exist. These integrals are important in applications dealing with Fourier transforms such as in the solution of integral equations of convolution type. We shall give approximations of these integrals in terms of finite sums of (2.20), (2.21), and (2.22) and (2.23) and we shall give bounds on the error of these approximations.

THEOREM 3.11 [46,49]: Let $f \in B_p(\mathcal{D}_d)$, $1 \leq p < \infty$, and let $\mathcal{E}(f,h)$ be defined as in (3.6). Then

$$(3.39) \qquad \mathcal{P}[\mathcal{E}(f,h)](x) = -\frac{1}{4\pi} \int_{\mathbb{R}} \left\{ \frac{[e^{i\pi x/h} + e^{-i\pi(t-id)/h}]\, f(t-id)}{(x-t-id)\sin\left[\frac{\pi}{h}(t-id)\right]} - \frac{[e^{i\pi x/h} + e^{i\pi(t+id)/h}]\, f(t+id)}{(x-t+id)\sin\left[\frac{\pi}{h}(t+id)\right]} \right\} dt$$

and

$$(3.40) \qquad \mathcal{H}[\mathcal{E}(f,h)](x) = -\frac{1}{2\pi} \int_{\mathbb{R}} \left\{ \frac{[\cos(\pi x/h) + e^{-i\pi(t-id)/h}]\, f(t-id)}{(t-x-id)\sin[\pi(t-id)/h]} + \frac{[\cos(\pi x/h) + e^{i\pi(t+id)/h}]\, f(t+id)}{(t-x+id)\sin[\pi(t+id)/h]} \right\} dt$$

Moreover, if $p=1$, then

$$\|P[E(f,h)]\|_\infty \leq \frac{1+\frac{1}{2}\,e^{-\pi d/h}}{2\pi d}\;\frac{N(f,\mathcal{D}_d)}{\sinh(\pi d/h)}\,, \tag{3.41}$$

$$\|H[E(f,h)\|_\infty \leq \frac{1+e^{-\pi d/h}}{2\pi d}\;\frac{N(f,\mathcal{D}_d)}{\sinh(\pi d/h)}\,; \tag{3.42}$$

while if $p=2$, then

$$\|P[E(f,h)]\|_\infty \leq \frac{1+\frac{1}{2}\,e^{-\pi d/h}}{2(\pi d)^{1/2}}\;\frac{N(f,\mathcal{D}_d)}{\sinh(\pi d/h)}\,; \tag{3.43}$$

$$\|H[E(f,h)]\|_\infty \leq \frac{1+e^{-\pi d/h}}{2(\pi d)^{1/2}}\;\frac{N(f,\mathcal{D}_d)}{\sinh(\pi d/h)}\,. \tag{3.44}$$

Furthermore, if $\|E(f,h)\|_2$ is bounded, then

$$\|P[E(f,h)]\|_2 = \|H[E(f,h)]\|_2 = \|E(f,h)\|_2\,. \tag{3.45}$$

In addition, if p is either 1 or 2, if f satisfies (3.5) on $\mathbb{R}$, and if $h = [\pi d/(\alpha N)]^{1/2}$, then there exists a constant C_1 depending only on f, d and α , such that

$$\left\| (Pf)(x) - \frac{1}{2i}\sum_{k=-N}^{N} f(kh)\left|\frac{e^{i\pi(x-kh)}-1}{\pi(x-kh)/h}\,,\right|\right\|_s \leq C_1 N^{1/2} e^{-(\pi d\alpha N)^{1/2}} \tag{3.46}$$

$$\| (Hf)(x) - i\sum_{k=-N}^{N} f(kh)\frac{\pi}{2h}(x-kh)S^2(0,1)\circ[\frac{x-kh}{2h}]\|_s \leq C_1 N^{1/2} e^{-(\pi d\alpha N)^{1/2}}\,, \tag{3.47}$$

for all $s \in [2,\infty]$.

We remark that if f satisfies (3.5), then so do $\mathcal{P}f$ and $\mathcal{H}f$. Consequently, we have (see (2.21) and (2.23)).

THEOREM 3.12: If $f \in B_p(\mathcal{D}_d)$, $p=1$ or 2, if f satisfies (3.5), and if $h = [\pi d/(\alpha N)]^{1/2}$, then there is a constant C_1, depending only on f, d and α, such that*

$$\left\| \mathcal{P}f - \sum_{k=-N}^{N} \left\{ \frac{1}{2} f(kh) + \frac{h}{2\pi i} \sum_{\substack{j=-N \\ j\neq k}}^{N} f(jh) \left| \frac{(-1)^{k-j}-1}{k-j} \right| S(k,h) \right\|_s \tag{3.48}$$

$$\leq C_1 \; N^{3/2} e^{-(\pi d \alpha N)^{1/2}} ,$$

and

$$\left\| \mathcal{H}f - \frac{i}{\pi} \sum_{k=-N}^{N} \left\{ \sum_{\substack{j=-N \\ j\neq k}}^{N} f(jh) \left| \frac{1-(-1)^{k-j}}{k-j} \right| \right\} S(k,h) \right\|_s \tag{3.49}$$

$$\leq C_1 \; N^{3/2} e^{-(\pi d \alpha N)^{1/2}} .$$

for all $s \in [2,\infty]$.

*These results are believed to be new.

3.7 Singularities on the Interval of Approximation.

In this section we consider the extension of the results (k), (ℓ), (m) and (n) of Theorem 2.1 to approximation in $B(\mathcal{D}_d)$. These results are particularly useful in the solution of singular integral equations in more than one dimension. These results are belived to be new.

THEOREM 3.13: Let $f \in B(\mathcal{D}_d)$, and let $\tau_k(a,\beta,h;x)$ be defined as in (2.27). Then for $1 \leqq p < \infty$, $1/p + 1/q = 1$, and $p(\beta+\frac{1}{2}) > \frac{1}{2}$

$$\left| \int_{\mathbb{R}} \frac{f(t)\,dt}{[(x-t)^2+a^2]^{\beta+1/2}} - \sum_{k=-\infty}^{\infty} f(kh)\tau_k(a,\beta,h;x) \right| \tag{3.50}$$

$$\leqq \pi^{1/2} a^{\frac{1}{\beta}-2(\beta+1/2)} d^{1/q} \left| \frac{\Gamma(P(\beta+\frac{1}{2})-1/2)}{\Gamma(P(\beta+1/2))} \right|^{1/p} \left| \frac{\Gamma(\frac{q-1}{2})}{\Gamma(\frac{q}{2})} \right|^{1/q} \frac{N(f,\mathcal{D}_d)}{\sinh(\pi d/h)}$$

Moreover, if f satisfies (3.5) and if $\tau_k(a,\beta,h)$ is defined as in (2.28) and (2.29) then there exists a constant C_1 depending only on f, d, α and a such that if $h = [\pi d/(\alpha N)]^{1/2}$,

$$\left| \int_{\mathbb{R}} \frac{f(t)\,dt}{[(x-t)^2+a^2]^{\beta+1/2}} - \sum_{k=-N}^{N} \left\{ \sum_{j=-N}^{N} f(jh)\tau_{k-j}(a,\beta,h) \right\} S(k,h)(x) \right| \tag{3.51}$$

$$\leqq C_1 N^{3/2} e^{-(\pi d\alpha N)^{1/2}}$$

In particular, if $a=0$ and $-\frac{1}{2} < \beta < 0$, then for f as in (3.50) ,

$$\left| \int_{\mathbb{R}} f(t)|x-t|^{-2\beta-1}dt - h^{-2\beta} \frac{\Gamma(-2\beta)\cos(\pi\beta)}{\pi} \sum_{k=-\infty}^{\infty} f(kh) \int_0^{\pi} t^{2\beta}\cos[(x-kh)t/h]dt \right| \tag{3.52}$$

$$\leqq \frac{\Gamma(-\beta)\Gamma(\frac{1}{2}+\beta)}{\pi^{1/2}2\pi d^{2+2\beta}} \frac{N(f,\mathcal{D}_d)}{\sinh(\pi d/h)}$$

and if f satisfies (3.5), then for $h = [\pi d/(\alpha N)]^{1/2}$

$$\left\| \int_{\mathbb{R}} f(t)|\cdot - t|^{-2\beta-1}\, dt - \frac{h^{-2\beta}\Gamma(-2\beta)\cos(\pi\beta)}{\pi} \sum_{k=-N}^{N} \left\{ \sum_{j=-N}^{N} f(jh) \int_0^{\pi} t^{2\beta}\cos[(k-j)t]dt \right\} \cdot \right.$$

(3.53)

$$\left. \cdot S(k,h) \right\|_{\infty} \leq C_1\, N^{3/2} e^{-(\pi d\alpha N)^{1/2}}$$

where C_1 depends only on f, β, α and d .

Equations (2.32) and (2.33) yield solutions to Laplace's equation in the upper half plane. Let u satisfy

(3.54)
$$u_{xx} + u_{yy} = 0 \text{ , } x \in \mathbb{R} \text{ , } y > 0$$

and either

(3.55)
$$\lim_{y \to 0^+} u(x,y) = f(x)$$

or

(3.56)
$$\lim_{y \to 0^+} \frac{\partial u(x,y)}{\partial y} = f(x)$$

<u>THEOREM 3.14:</u> Let u be the solution of the Dirichlet problem (3.54), (3.55), where f satisfies the condition of Theorem 3.2. Then

$$\left\| u(\cdot,y) - \frac{h}{\pi} \sum_{k=-\infty}^{\infty} f(kh) \left\{ \frac{y(1-e^{-\pi y/h})\cos[\pi(\cdot - kh)/h] + (\cdot - kh)e^{-\pi y/h}\sin[\pi(\cdot - kh)/h]}{(\cdot - kh)^2 + y^2} \right\} \right\|_p$$

(3.57)

$$\leq \| E(f,h) \|_p$$

where $\|E(f,h)\|_p$ is defined and bounded as in Theorem 3.2. Moreover, if f also satisfies Corollary (3.3), then by choosing h and s as in Corollary 3.3,

$$\| u(\cdot,y) - \frac{h}{\pi} \sum_{k=-N}^{N} \left\{ \sum_{j=-N}^{N} f(jh) \frac{y(1-e^{-\pi y/h})(-1)^{k-j}}{(k-j)^2 h^2 + y^2} \right\} S(k,h) \|_s \tag{3.58}$$

$$\leq C_1 \, N^{3/2} e^{-(\pi d \alpha N)^{1/2}}$$

where C_1 depends only on f, d and α .

<u>THEOREM 3.15:</u> Let $f \in B(\mathcal{D}_d)$, and let

$$M(a,\ f,\ \mathcal{D}_d) = \int_{\mathbb{R}} [|f(t+id^-)| + |f(t-id^-)|] \log[(t-a)^2 + d^2] dt < \infty \ . \tag{3.59}$$

for all finite $a \in \mathbb{R}$. Then the function $u = u(x,y)$,

$$u(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \log[(x-t)^2 + y^2] f(t) dt \tag{3.60}$$

which solves the Neumann problem (3.54), (3.56) satisfies

$$\| u(\cdot,y) - \frac{h}{2\pi} \sum_{k=-\infty}^{\infty} f(kh) \Big\{ -\gamma - \log\pi/h + [1-\cos\{\pi(\cdot-kh)/h\}] \log[\frac{(\cdot-kh)^2+y^2}{(\cdot-kh)^2}]$$

$$+ \int_0^{\pi} \frac{1-(2-e^{-ty/h})\cos[(\cdot-kh)t/h]}{t} \Big\} dt \|_{\infty} \ . \tag{3.61}$$

$$\leq \frac{M(a,\ f,\ \mathcal{D}_d) e^{-\pi d/h} + 4(h/d) N(f, \mathcal{D}_d)}{8\pi \sinh(\pi d/h)}$$

Moreover, if f satisfies (3.5), and if

$$\int_{\mathbb{R}} f(t) dt = 0 \quad , \tag{3.62}$$

then for $h = [\pi d/(\alpha N)^{1/2}]$,

$$\Big\| u(\cdot,y)-\frac{h}{2\pi}\sum_{k=-N}^{N}\Big\{\sum_{\substack{j=-N\\ j\neq k}}^{N} f(jh)[1-(-1)^{k-j}]\log[\frac{(k-j)^2+y^2/h^2}{(k-j)^2}]$$

(3.63)
$$+\int_0^{\pi}\frac{1-(2-e^{-ty/h})\cos[(k-j)t]}{t}dt\Big\}S(k,h)\Big\|_\infty$$

$$\leq C_1 N^{3/2}e^{-(\pi d\alpha N)^{1/2}}$$

where C_1 depends only on f, d and α. In particular, if $y=0$, (3.63) reduces to

$$\Big\|\int_{\mathbb{R}}\log|\cdot-t|f(t)dt-\frac{h}{2\pi}\sum_{k=-N}^{N}\Big\{\sum_{\substack{j=-N\\ j\neq k}}^{N} f(jh)\int_0^{\pi}\frac{1-\cos[(k-j)t]}{t}\,dt\Big\}S(k,h)\Big\|_\infty$$

(3.64)
$$\leq C_1 N^{3/2}e^{-(\pi d\alpha N)^{1/2}}$$

4. FORMULAS OVER FINITE, SEMI-INFINITE INTERVALS AND CONTOURS.

In this section we shall consider the extension of some of the formulas of the previous section to finite and semi infinite intervals, and more generally, to contours [45,49].

The success of the methods of the previous section for functions in $\mathcal{D}_d$ suggests that a problem over an arbitrary interval be transformed into one over $(-\infty,\infty)$ in such a way that the conditions of the theorems of the previous section are satisfied. In this way, some formulas over contours other than $(-\infty,\infty)$ may be obtained directly. However, not all formulas transform directly, and we must make certain simple "adjustments" in order to make direct transformation possible.

The following definition is fundamental for the remainder of the paper.

DEFINITION 4.1: Let $\mathcal{D}$ be a simple connected domain, with boundary $\partial\mathcal{D}$, let a and $b \neq a$ be points of $\partial\mathcal{D}$ and let $\mathcal{D}_d$ be defined as in (3.1). Let ϕ be a conformal map of $\mathcal{D}$ onto $\mathcal{D}_d$, such that $\phi(a) = -\infty$, $\phi(b) = \infty$. Let $\psi = \phi^{-1}$ denote the inverse map, and set

$$\Gamma = \{\psi(x) : -\infty \leqq x \leqq \infty\} \tag{4.1}$$

Given ϕ and ψ, we denote by $z_k = z_k(h)$ the points

$$z_k = \psi(kh), \quad k=0,\pm 1,\pm 2,\ldots \quad . \tag{4.2}$$

Let $B(\mathcal{D})$ denote the family of all functions F that are analytic in $\mathcal{D}$, such that

$$\int_{\psi(u+L)} |F(z)dx| \to 0 \quad \text{as} \quad u \to \pm\infty \tag{4.3}$$

where

$$L = \{iy : y \text{ is real} \ , \ |y| \leq d\} \quad , \tag{4.4}$$

and such that

$$N(F,\mathcal{D}) \equiv \lim_{C_1 \to \partial\mathcal{D}} \inf_{C_1 \subset \mathcal{D}} \int_{C_1} |F(z)dz| < \infty \tag{4.5}$$

We remark that if $F \in B(\mathcal{D})$, then f defined by

$$f = [F \circ \psi]\psi' \tag{4.6}$$

is in $B(\mathcal{D}_d)$ as defined in Definition 3.1.

Let us next give four commonly used transformations ϕ and the corresponding inverse functions ψ , corresponding intervals $\Gamma = [0,1],[-1,1]$ and $[0,\infty]$.

<u>EXAMPLE 4.1:</u> $\Gamma = [0,1]$. In this case

$$\mathcal{D} = \{z : |\arg \frac{z}{1-z}| < d\} \ , \tag{4.7}$$

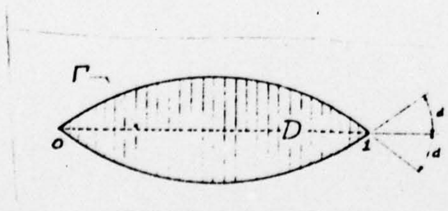

FIGURE 4.1 The Region $\mathcal{D}$ of Ex. 4.1.

The boundary of $\mathcal{D}$ consists of two circular arcs which intersect with angle $2d$ at 0 and at 1 .

The function ϕ , the inverse function ψ and the points z_k are given by

(4.8)
$$w = \phi(z) = \log\frac{z}{1-z} \Longleftrightarrow z = \psi(w) = \frac{1}{2}+\frac{1}{2}\tanh\frac{w}{2}$$
$$z_k = \frac{1}{2}+\frac{1}{2}\tanh(kh/2) \ , \ k=0,\pm1,\pm2,\ldots \quad .$$

EXAMPLE 4.2: $\Gamma = [-1,1]$. In this case

(4.9)
$$\mathcal{D} = \{z : |\arg(\frac{1+z}{1-z})| < d\} \quad .$$

The functions ϕ and ψ and the points z_k are given by

(4.10)
$$w = \phi(z) = \log(\frac{1+z}{1-z}) \Longleftrightarrow z = \psi(w) = \tanh\frac{w}{2}$$
$$z_k = \tanh(kh/2) \ , \ k=0,\pm1,\pm2,\ldots \quad .$$

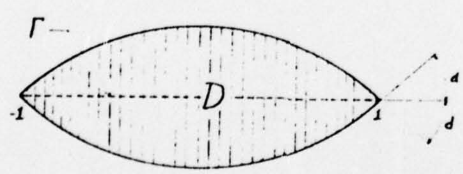

FIGURE 4.2 The Region $\mathcal{D}$ of Ex. 4.2.

We shall give two examples for the case $\Gamma = [0,\infty]$. The first of these is useful in the case when the function f to be approximated is analytic in a sector, $|\arg z| < d$, while the second is useful if f is analytic only in a strip of width $2d$ symmetric about the real axis (more precisely, in the region $\mathcal{D}$ of Ex. 4--see Figure 4.4).

EXAMPLE 4.3: $\Gamma = [0,\infty]$. In this case $\mathcal{D}$ is the sector

$$\mathcal{D} = \{z : |\arg z| < d\} \quad . \tag{4.11}$$

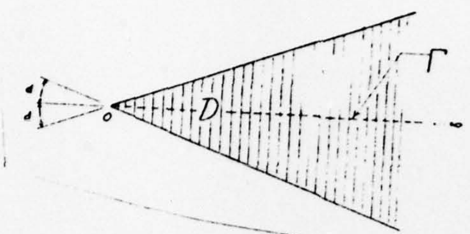

FIGURE 4.3 The Region $\mathcal{D}$ of Ex. 4.3.

The functions ϕ,ψ and the points z_k are given by

(4.12)

$$w = \phi(z) = \log z \Longleftrightarrow z = \psi(w) = e^w$$

$$z_k = e^{kh} \text{ , } k=0,\pm1,\pm2,\ldots \quad .$$

EXAMPLE 4.4 [22]: $\Gamma = [0,\infty]$. In this case

$$\mathcal{D} = \{z : |\arg \sinh(z)| < d\} \text{ , } 0 < d \leq \pi/2 \tag{4.13}$$

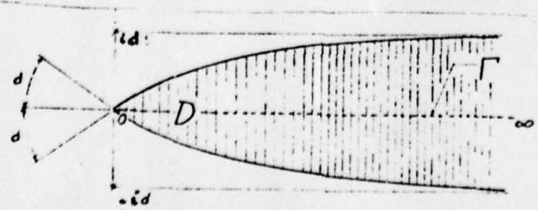

FIGURE 4.4 The Region $\mathcal{D}$ of Ex. 4.4.

The functions ϕ,ψ and the points z_k are given by

$$w = \phi(z) = \log[\sinh z] \Longleftrightarrow z = \psi(w) = \log[e^w + \sqrt{1+e^{2w}}\,]$$

$$(4.14) \qquad z_k = \log[e^{kh} + \sqrt{1+e^{2kh}}\,]\ ,\ k=0,\pm 1,\pm 2,..$$

$$= e^{kh} - \frac{1}{6}e^{3kh} + \frac{3}{40}e^{5kh} - \frac{5}{112}e^{7kh} + \ldots, e^{kh} < .1$$

The expansion in (4.14) is preferred if $e^{kh} < .1$ since the formula $z_k = \log[e^{kh} + \sqrt{1+e^{2kh}}]$, while mathematically exact, is computationally inaccurate for small e^{kh} . In applications, the accurate computation of $f(z_k)$, where $f(z_k)$ is to be approximated, is important, especially near a singularity of f (see Sec. 8.2).

We shall next review the known results corresponding to those in Sec. 3 which may be extended to approximation over a contour Γ as described in Def. 4.1, albeit some minor modifications. These modifications are described in the theorems, as we shall present them.

The condition (3.5) takes on a simple general form: let g be defined on Γ , and let

$$(4.15) \qquad |g(x)| \leq Ce^{-\alpha|\phi(x)|}$$

for all $x \in \Gamma$, where C and α are positive constants.

Two identities play an important role in obtaining all of our bounds. These are described in the following theorem.

<u>THEOREM 4.2 [49]:</u> Let $F \in B(\mathcal{D})$. Then the identity

$$\frac{F(x)}{\phi'(x)} - \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{\phi'(z_k)} S(k,h)\circ\phi(x)$$

(4.16)

$$= \frac{\sin[\pi\phi(x)/h]}{2\pi i} \int_{\partial\mathcal{D}} \frac{F(z)dz}{[\phi(z)-\phi(x)]\sin[\pi\phi(z)/h]}$$

is valid for all $x \in \Gamma$. Moreover

$$\int_{\Gamma} F(x)dx - h \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{\phi'(z_k)}$$

(4.17)

$$= \frac{1}{2} \int_{\partial\mathcal{D}} \frac{\exp[\frac{i\pi\phi(z)}{h} \operatorname{sgn} \operatorname{Im}\phi(z)]}{\sin[\pi\phi(z)/h]} F(z)dz \quad .$$

We remark that (4.17) is obtained from (4.16) by multiplying (4.16) by $\phi'(x)$ and integrating over Γ .

4.1 Interpolation over Γ .

THEOREM 4.3 [45,49]: Let $\phi' F \in B(\mathcal{D})$. Then

$$(4.18) \qquad \left| F(x) - \sum_{k=-\infty}^{\infty} F(z_k) S(k,h) \circ \phi(x) \right| \leqq \frac{N(\phi' F, \mathcal{D})}{2\pi d \sinh(\pi d/h)} .$$

for all $x \in \Gamma$. Moreover, if F is bounded on Γ by the right-hand side of (4.15), then by taking $h = [\pi d/(\alpha N)]^{1/2}$

$$(4.19) \qquad \left| F(x) - \sum_{k=-N}^{N} F(z_k) S(k,h) \circ \phi(x) \right| \leqq C_1 N^{1/2} e^{-(\pi d \alpha N)^{1/2}}$$

for all $x \in \Gamma$, where C_1 depends only on F, d and α .

It may happen that $F \in B(\mathcal{D})$, but $\phi' F \notin B(\mathcal{D})$. If the limits

$$(4.20) \qquad \begin{aligned} \lim_{x \to a} F(x) &= F(a) \\ \lim_{x \to b} F(x) &= F(b) \end{aligned}$$

exists and are bounded, where the limits are taken along Γ , then it may also be the case that $\phi' G \in B(\mathcal{D})$, where

$$(4.21) \qquad G = F - \frac{e^{-\frac{1}{2}\phi}}{e^{\frac{1}{2}\phi} + e^{-\frac{1}{2}\phi}} F(a) - \frac{e^{\frac{1}{2}\phi}}{e^{\frac{1}{2}\phi} + e^{-\frac{1}{2}\phi}} F(b)$$

This device is often useful in applications.

Let us next illustrate the formula (4.19) and (4.21) for the case of the transformations in Examples 4.1 to 4.4.

EXAMPLE 4.5: For the case when $\Gamma = [0,1]$, let F be analytic and bounded in the domain $\mathcal{D}$ of (4.7). On $[0,1]$, the condition (4.15) becomes

$$|F(x)| \leqq C\, x^{\alpha}(1-x)^{\alpha} \quad . \tag{4.22}$$

If F satisfies (4.22) on $[0,1]$, then by taking $h = [\pi d/(\alpha N)]^{1/2}$, (4.19) becomes

$$\left| F(x) - \frac{h}{\pi} \sin\left\{\frac{\pi}{h} \log \frac{x}{1-x}\right\} \sum_{k=-N}^{N} \frac{(-1)^k F(z_k)}{\log\left(\frac{x}{1-x}\right) - kh} \right| \leqq C_1 N^{1/2} e^{-(\pi d\alpha N)^{1/2}} \tag{4.23}$$

where z_k is given by (4.8), and where C_1 depends on F, d and α . If F does not vanish on 0 or at 1 , the function G of (4.21) may, provided that the limits (4.20) exist. The function G takes the form

$$G(x) = F(x) - (1-x)F(0) - xF(1) \quad . \tag{4.24}$$

We remark that if $F \in B(\mathcal{D})$ and if $F \in \mathrm{Lip}_{\alpha}(\overline{\mathcal{D}})$, where $\overline{\mathcal{D}}$ denotes the closure of $\mathcal{D}$, then $\phi' G \in B(\mathcal{D})$ and G satisfies (4.22) on $[0,1]$ [36] . The formula (4.23) does a good job of interpolating functions such as

$F(x) = x^{1/3}(1-x)^{-1/2}\log x$, or $F(x) = \sin(\pi x)\log(1-x)$, etc. on $[0,1]$.

The case of $\Gamma = [-1,1]$ is similar to that of $\Gamma = [0,1]$.

EXAMPLE 4.6: Consider the case of $\mathcal{D}$ as in Ex. 4.3, with $\Gamma = [0,\infty]$. Let F be analytic in the sector (4.11). On $[0,\infty]$, let F satisfy

$$|F(x)| \leqq \begin{cases} Cx^{\alpha} & \text{if} \quad 0 \leqq x \leqq 1 \\ Cx^{-\alpha} & \text{if} \quad 1 \leqq x \leqq \infty \end{cases} , \tag{4.25}$$

a condition which is equivalent to (4.15). Then by taking $h = [\pi d/(\alpha N)]^{1/2}$, (4.19) becomes

$$
|F(x) - \frac{h}{\pi} \sin\{\frac{\pi}{h} \log x\} \sum_{k=-N}^{N} \frac{(-1)^k F(e^{kh})}{\log x - kh}| \leq C_1 N^{1/2} e^{-(\pi d \alpha N)^{1/2}} \tag{4.26}
$$

where C_1 depends only on F, d and α. If F does not vanish at 0 or at ∞ and if the limits (4.20) exist, then we may be able to effectively apply (4.26) to G (see Eq. (4.21)) where

$$
G(x) = F(x) - \frac{1}{1+x} F(0) - \frac{x}{1+x} F(\infty) \quad . \tag{4.27}
$$

The formula (4.26) does an accurate job of interpolating functions F such as $F(x) = x^{2/3}(\log x)/(1+x)$, or for $F(x) = x^{5/2}e^{-x}\sin x/2$, etc.

<u>EXAMPLE 4.7</u>: Let us again take $\Gamma = [0,\infty]$, for the case of Ex. 4.4. For this case the condition (4.15) becomes

$$
|F(x)| \leq \begin{cases} C\, x^{\alpha} & \text{if} \quad 0 \leq x \leq 1 \\ C\, e^{-\alpha x} & \text{if} \quad 1 \leq x \leq \infty \end{cases} \quad . \tag{4.28}
$$

If $\phi' F \in B(\mathcal{D})$ and if F satisfies (4.28) on $[0,\infty]$, then by taking $h = [\pi d/(\alpha N)]^{1/2}$, we get

$$
\left| F(x) - \frac{h}{\pi} \sin\{\frac{\pi}{h} \log[\sinh x]\} \sum_{k=-N}^{N} \frac{(-1)^k F(z_k)}{\log[\sinh x] - kh} \right| \leq C_1 N^{1/2} e^{-(\pi d \alpha N)^{1/2}} \tag{4.29}
$$

for all $x \in [0,\infty]$, where z_k is defined in (4.14) and where C_1 depends only on F, d and α . If F is analytic in $\mathcal{D}$ but if F does not vanish at 0 or at ∞ , then the function G may satisfy the conditions of Theorem 4.3 well as (4.28), where

$$(4.30) \qquad G(x) = F(x) - \frac{1}{1+\sinh x} F(0) - \frac{\sinh x}{1+\sinh x} F(\infty) \quad .$$

The formula (4.29) does well at interpolating functions over $[0,\infty]$ which may be oscillatory on $(0,\infty)$, but which may have a singularity at $x=0$. For example, (4.29) does well at interpolating $F(x) = x^{\alpha}\log[1-(\frac{\sin x}{x})^2]\, e^{-\alpha x}$, or $F(x) = x^{4/5}e^{-x}$.

4.2 Quadrature over Γ.

Eq. (4.17) yields the following theorem.

THEOREM 4.4 [45]: Let $F \in B(\mathcal{D})$. Then

$$(4.31) \qquad \left| \int_\Gamma F(x)dx - h \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{\phi'(z_k)} \right| \leq \frac{e^{-\pi d/h}}{2 \sinh(\pi d/h)} N(F,\mathcal{D}) \quad .$$

Moreover, if F/ϕ' satisfies (4.15) on Γ, then by taking $h = [2\pi d/(\alpha N)]^{1/2}$,

$$(4.32) \qquad \left| \int_\Gamma F(x)dx - h \sum_{k=-N}^{N} \frac{F(z_k)}{\phi'(z_k)} \right| \leq C_1 e^{-(2\pi d\alpha N)^{1/2}}$$

where C_1 depends only on F, d and α.

EXAMPLE 4.8: Let $\Gamma = [-1,1]$, and let $F \in B(\mathcal{D})$ where $\mathcal{D}$ is defined in (4.9). On $[-1,1]$, let

$$(4.33) \qquad |F(x)| \leq C\,(1-x^2)^{\alpha-1} \quad , \alpha > 0 \quad , C > 0 \; ,$$

a condition equivalent to F/ϕ' satisfying (4.15) on $\Gamma = [-1,1]$. Then by taking $h = [2\pi d/(\alpha N)]^{1/2}$,

$$(4.34) \qquad \left| \int_{-1}^{1} F(x)dx - h \sum_{k=-N}^{N} \frac{2e^{kh}}{(1+e^{kh})^2} f\left(\frac{e^{kh}-1}{e^{kh}+1}\right) \right| \leq C_1 e^{-(2\pi d\alpha N)^{1/2}}$$

where C_1 depends only on F, d and α. The formula (4.15) yields accurate results for the integration of function F such as $F(x)=(1-x)^{-1/3}(1+x)^{-3/5}\log(1-x)$ or $F(x)=(1-x)^{-4}\exp\{-2/(1-x)\}$.

EXAMPLE 4.9: Let $\Gamma = [0,\infty]$, and let $F \in B(\mathcal{D})$ where $\mathcal{D}$ is defined in (4.11). On $[0,\infty]$, let

$$|F(x)| \leqq \begin{cases} C\, x^{\alpha-1} & , \quad 0 \leqq x \leqq 1 \\ C\, x^{-\alpha-1} & , \quad 1 \leqq x \leqq \infty \end{cases} \quad , \tag{4.35}$$

a condition which is equivalent to F/ϕ' satisfying (4.15). Then by taking $h = [2\pi d/(\alpha N)]^{1/2}$

$$\left| \int_0^\infty F(x)dx - h \sum_{k=-N}^{N} e^{kh} F(e^{kh}) \right| \leqq C_1\, e^{-(2\pi d\alpha N)^{1/2}} \tag{4.36}$$

where C_1 depends only on F, d and α. The formula (4.36) does an accurate job of integrating function F such as $F(x) = x^{\alpha-1}/(1+x)^{2\alpha}$, or $F(x) = x^{-3/2}\sin(x/2)e^{-x}$.

EXAMPLE 4.10 [22]: Let $\Gamma = [0,\infty]$, and let $F \in B(\mathcal{D})$, where $\mathcal{D}$ is defined in (4.14). On $[0,\infty]$, let

$$|F(x)| \leqq \left\{ \begin{array}{ll} C\, x^{\alpha-1} & \text{if} \quad 0 \leqq x \leqq 1 \\ C\, e^{-\alpha x} & \text{if} \quad 1 \leqq x \leqq \infty \end{array} \right\} \quad \alpha > 0 \tag{4.37}$$

a condition which is equivalent to F/ϕ' satisfying (4.15). Then by taking $h = [2\pi d/(\alpha N)]^{1/2}$,

$$\begin{aligned} &\left| \int_0^\infty F(x)dx - h \sum_{k=-N}^{N} \frac{1}{\sqrt{1+e^{-2kh}}}\, F\left\{ \log[e^{kh} + \sqrt{1+e^{2kh}}\,] \right\} \right| \\ &\qquad \leqq C_1\, e^{-(2\pi d\alpha N)^{1/2}} \end{aligned} \tag{4.38}$$

where C_1 depends only on F, d and α. The formula (4.38) does an accurate job of integrating functions F such as $F(x) = x^{-1/2}\log[1-\frac{\sin x}{x}]e^{-x/2}$, $F(x) = x^{-2/7}e^{-x^2}$, or $F(x) = x^{-5}\exp\{-[(x-5)^2+2]^{1/2}-1/x^2\}\sin(3x)$.

4.3 Approximation of Derivatives on Γ .

Except on $\mathbb{R}$, the formula (4.18) is not useful for accurately approximating derivatives of F on Γ , since the terms $\phi' S(k,h)\circ\phi$ are unbounded on Γ . Then to get a formula for approximating $f^{(m)}$ on Γ it becomes necessary to modify (4.18) by introducing a "nullifier" function g with the property that

$$\left(\frac{d}{dx}\right)^n \left\{g(x)S(k,h)\circ\phi(x)\right\}$$

is bounded on Γ , for $n=0,1,\dots,m$. Upon replacing F in (4.16) by $F\phi'/g$ we get

THEOREM 4.5 [25]: Let $F\phi'/g \in B(\mathcal{D})$. Then for all $x \in \Gamma$,

$$F(x) - \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{g(z_k)} g(x)S(k,h)\circ\phi(x)$$

(4.39)

$$= \frac{g(x)\sin[\pi\phi(x)/h]}{2\pi i} \int_{\partial\mathcal{D}} \frac{F(z)\phi'(z)/g(z)\ dz}{[\phi(z)-\phi(x)]\sin[\pi\phi(z)/h]}$$

Let

$$\left| \left(\frac{d}{dx}\right)^n \frac{g(x)\ \sin[\pi\phi(x)/h]}{\phi(z)-\phi(x)} \right| \leq C_2 h^{-n} \tag{4.40}$$

for $n=0,1,\dots,m$, for all $x \in \Gamma$ and $z \in \partial\mathcal{D}$, where C_2 is a constant depending only on m, g and F . Then there exists a constant C_3 , depending only on m, g, d and F , such that for all $x \in \Gamma$,

$$\left|F^{(n)}(x) - \left(\frac{d}{dx}\right)^n \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{g(z_k)} g(x)S(k,h)\circ\phi(x)\right|$$

(4.41)

$$\leq C_3 h^{-n} e^{-\pi d/h} \quad , \ n=0,1,\dots,m \quad .$$

Moreover, if F/g satisfies (4.15) on Γ, then by taking $h = [\pi d/(\alpha N)]^{1/2}$, then for all $x \in \Gamma$

$$\left|F^{(n)}(x) - \left(\frac{d}{dx}\right)^n \sum_{k=-N}^{N} \frac{F(z_k)}{g(z_k)} g(x)S(k,h)\circ\phi(x)\right|$$

(4.42)

$$\leq C_1 N^{\frac{n+1}{2}} e^{-(\pi d\alpha N)^{1/2}} \quad , n=0,1,\ldots,m \quad ,$$

where C_1 depends only on m, g d, F and α.

The function g takes on different forms for different values of ϕ although $g(x) = [1/\phi'(x)]^m$ is usually satisfactory. In addition Theorem 4.5 presupposes that all derivatives of F vanish at a certain rate as $x \to a$ and as $x \to b$ along Γ. We shall present special forms for g as well as procedures for making a number of derivatives of F vanish at a and at b in the examples which follow.

<u>EXAMPLE 4.11 [25]:</u> Let $\Gamma = [-1,1]$, and let $\mathcal{D}$ be defined as in (4.9). Let us take

(4.43) $$g(x) = (1-x^2)^m$$

and let us assume that $[F\phi'/g] \in B(\mathcal{D})$ where $F(x)\phi'(x)/g(x) = 2F(x)(1-x^2)^{-m-1}$. Furthermore, on $[-1,1]$, let us assume that

(4.44) $$\left|\frac{F(x)}{g(x)}\right| = \left|\frac{F(x)}{(1-x^2)^m}\right| \leq C(1-x^2)^\alpha$$

a condition on F/g equivalent to (4.15). Then by taking $h = [\pi d/(\alpha N)]^{1/2}$

$$\left|F^{(n)}(x) - \left(\frac{d}{dx}\right)^n \left\{ \frac{h}{\pi} (1-x^2)^m \sin\left[\frac{\pi}{h} \log\left(\frac{1+x}{1-x}\right)\right] \sum_{k=-N}^{N} \frac{(-1)^k F(z_k)/(1-z_k)^m}{\log\left[\frac{1+x}{1-x}\right] - kh} \right\}\right|$$

(4.45)

$$\leq C_1 N^{\frac{n+1}{2}} e^{-(\pi d\alpha N)^{1/2}} \quad , \quad n=0,1,\ldots,m \quad ,$$

for all $x \in \Gamma$. More generally, let $F^{(m)}$ be analytic and bounded on $\mathcal{D}$ and $F^{(m)} \in \mathrm{Lip}_{\alpha}(\bar{\mathcal{D}})$, where $\bar{\mathcal{D}}$ denotes the closure of $\mathcal{D}$. In this case (4.44) may not be satisfied. However, the function

$$G = F - p_m \tag{4.46}$$

satisfies (4.44) as well as all of the other conditions required in (4.45), where p_m is constructed as follows. Set

$$p_0(x) = a_0 \frac{1-x}{2} + b_0 \frac{1+x}{2} \tag{4.47}$$

where

$$a_0 = f(-1) \ , \ b_0 = f(1) \tag{4.48}$$

and

$$p_{k+1}(x) = p_k(x) + a_{k+1}\left(\frac{1-x}{2}\right)^{k+2}\left(\frac{1+x}{2}\right)^{k+1} + b_{k+1}\left(\frac{1-x}{2}\right)^{k+1}\left(\frac{1+x}{2}\right)^{k+2} \ , \tag{4.49}$$

where

$$\begin{aligned} a_{k+1} &= \frac{2^{k+1}}{(k+1)!} \quad [f^{(k+1)}(-1) - p_k{}^{(k+1)}(-1)] \\ b_{k+1} &= \frac{(-2)^{k+1}}{(k+1)!} \quad [f^{(k+1)}(-1) - p_k{}^{(k+1)}(1)] \quad . \end{aligned} \tag{4.50}$$

<u>EXAMPLE 4.12 [23]:</u> Let $\Gamma = [0,\infty]$, and let F be analytic in the region $\mathcal{D}$ of Eq. (4.11). Let us assume that $F^{(m)}$ exists on $[0,\infty]$, let us take $g(x) = x^m$, and let us assume that $\phi' F/g \in B(\mathcal{D})$, where $[\phi' F/g](x) = F(x)/x^{m+1}$. Furthermore, on $[0,\infty]$, let us assume that

$$\left|\frac{F(x)}{x^n}\right| \leqq \begin{cases} C\,x^{\alpha} & , \quad 0 \leqq x \leqq 1 \\ C\,x^{-\alpha} & , \quad 1 \leqq x \leqq \infty \quad , \quad n = 0, 1, \ldots, m\ . \end{cases} \tag{4.51}$$

a condition on F/g equivalent to (4.15). Then by choosing $h = [\pi d/(\alpha N)]^{1/2}$,

$$\begin{aligned} &\left| F^{(n)}(x) - \left(\frac{d}{dx}\right)^n \frac{h}{\pi} \left\{ x^m \sin[\tfrac{\pi}{h} \log x] \sum_{k=-N}^{N} \frac{(-1)^k F(e^{kh}) e^{-mkh}}{\log x - kh} \right\} \right| \\ &\qquad \leqq C_1 N^{\frac{n+1}{2}} e^{-(\pi d \alpha N)^{1/2}} \quad , \ n=0,1,\ldots,m \end{aligned} \tag{4.52}$$

for all $x \in [0,\infty]$, where C_1 depends only on F, d, m and α . More generally if $F^{(m)} \in B(\mathcal{D})$ and if $F^{(m)} \in \mathrm{Lip}_{\alpha}(\overline{\mathcal{D}})$, then F may not satisfy (4.51). However, the function G usually does satisfy all of the requirements of (4.52), where

$$\begin{aligned} G(x) &= F(x) - e^{-x} \sum_{k=0}^{m} a_k x^k \\ a_0 &= F(0) \\ a_k &= [F^{(k)}(0) - \sum_{j=0}^{k-1} \frac{(-1)^{k-j} k!}{(k-j)!} a_j] \ , \ k=1,2,\ldots,m \quad . \end{aligned} \tag{4.53}$$

4.4 Approximation of the Indefinite Integral on Γ .

We shall give a general formula for approximating

$$I(x) = \int_a^x F(t)dt \quad , \quad x \in \Gamma \quad , \tag{4.54}$$

in Theorem 4.6 below. The special forms of this formula for the cases of Examples 4.1 to 4.4 are omitted, since these special cases are simply obtained by direct substitution. The results of this formula are especially suited to the solution of linear initial value problems and to some linear Volterra integral equations. The solutions of these equations may have a singularity or a boundary layer at one or both end-points of Γ .

THEOREM 4.6 [24]: Let $F \in B(\mathcal{D})$, and along Γ , let

$$\left|\frac{F(x)}{\phi'(x)}\right| \leq C\, e^{-\alpha'|\phi(x)|} \tag{4.55}$$

where C and α' are positive constants. Let $0 < \beta < \pi/d$, and let $G \in B(\mathcal{D})$, where

$$G(x) = \int_a^x F(t)dt - \frac{e^{\frac{1}{2}\beta\phi(x)}}{e^{\frac{1}{2}\beta\phi(x)} + e^{\frac{1}{2}\beta\phi(x)}} \int_a^b F(t)dt \tag{4.56}$$

Let

$$\alpha = \min(\alpha',\beta) \quad . \tag{4.57}$$

Then for $h = [\pi d/(\alpha N)]^{1/2}$,

$$(4.58)\qquad \Bigg| \int_a^x F(t)dt - \frac{e^{\frac{1}{2}\beta\phi(x)}}{e^{\frac{1}{2}\beta\phi(x)} + e^{-\frac{1}{2}\beta\phi(x)}} \int_\Gamma F(t)dt$$

$$- h \sum_{k=-N}^{N} \left\{ \sum_{j=-N}^{N} \sigma_{k-j} \left[\frac{F(z_j)}{\phi'(z_j)} - \frac{\beta}{(e^{\frac{1}{2}\beta jh} + e^{-\frac{1}{2}\beta jh})^2} \int_\Gamma F(t)dt \right] \right\} \cdot S(k,h)\circ\phi(x) \Bigg|$$

$$\leqq C_1 N e^{-(\pi d\alpha N)^{1/2}}$$

for all $x \in \Gamma$, where σ_k is defined in (2.15), and where C_1 depends only on F, d and α .

In applications the integral $\int_\Gamma F(t)dt$ is approximated by means of formula (4.32), i.e., by $h \sum_{k=-N}^{N} F(z_k)/\phi'(z_k)$.

For example, in applications, the formula (4.58) may be used to approximate an integral such as $\int_0^x t^{-2/3}(\log t)(1-t)^{-5/4}dt$, $x \in [0,1]$.

4.5 Singular Integrals on Γ .

In this section we consider the approximation of the integrals

$$(\mathcal{H}F)(x) \equiv \text{P.V.} \int_\Gamma \frac{F(t)}{t-x}\,dt \ , \ x \in \Gamma \ ; \tag{4.59}$$

$$(\mathcal{U}_{\beta,y}F)(x) \equiv \int_\Gamma [(x-t)^2+y^2]^{-\beta-1/2} F(t)dt \ , \ x \in \Gamma \ , \ y > 0 \ , \ \beta \geqslant -1/2 \ , \tag{4.60}$$

$$(\mathcal{V}_\beta F)(x) \equiv \int_\Gamma |x-t|^{\beta-1} F(t)dt \ , \ x \in \Gamma \ , \ \beta > 0 \ ; \tag{4.61}$$

$$(\mathcal{W}F)(x) \equiv \int_\Gamma \log|x-t| \ F(t)dt \ , \ x \in \Gamma \ . \tag{4.62}$$

The results of this section are believed to be new.

Although the last three of these integrals can be evaluated via the method in Sec. 4.2, by splitting up each integral as an integral from a to x plus an integral from x to b , the methods developed in this section are more efficient, since we derive explicit approximate expressions that are valid for all $x \in \Gamma$. This increased efficiency is especially important in the solution of integral equations, where the major difficulty is the evaluation of many singular inner product integrals.

Although the case of $\mathcal{H}F$ is done fairly generally, due to its importance, the procedure for the case of the remaining integrals is illustrated only for special intervals. We have also left out the case of $\int_\Gamma F(t)\log[|x-t|^2+|y|^2]\,dt$, since formulas for this case can readily be derived by combining the procedures for (4.62) and (4.60).

A simple treatment is required for each of the above integrals, in order to be able to use a suitable formula in Sec. 3, after transformation from $\mathcal{D}$ to $\mathcal{D}_d$. Denoting an arbitrary one of the integrals (4.59) to (4.62) by $\mathcal{T}F$, we assume that F is continuous and bounded on Γ, and we first set

$$F = \mathcal{L}F + E \quad , \tag{4.63}$$

where $\mathcal{L}F$ is a simple, suitable explicit interpolation of F at the end-points a and b of Γ. Then E is continuous on Γ, and $E(a) = E(b) = 0$. We then construct a simple "polynomial" p, defined on Γ, such that $p(a) = p(b) = 0$, such that $P = \mathcal{T}p$ can be explicitly expressed, and such that the function G defined by

$$G = E - p \tag{4.64}$$

satisfies:

$$\begin{gathered} \mathcal{T}G \text{ exists on } \Gamma \ ; \\ (\mathcal{T}G)(x) \to 0 \quad \text{on } x \to a \text{ or } b \text{ , along } \Gamma \ , \\ G \text{ is analytic in } \mathcal{D} \ . \end{gathered} \tag{4.65}$$

At this point we can apply a suitable formula from Sec. 3 to $\mathcal{T}G$, after transformation from $\mathcal{D}$ to $\mathcal{D}_d$.

Let us illustrate the above outlined procedure on examples of the approximation of (4.59) - (4.62).

<u>EXAMPLE 4.13:</u> <u>The Hilbert Transform over a Finite Γ.</u> Let us make the following assumptions:

(i) $F \in B(\mathcal{D})$, where $\mathcal{D}$ is bounded, and Γ is finite. Let a and b denote the end-points of Γ;

(ii) $F \in Lip_\alpha(\overline{\mathcal{D}})$, where $0 < \alpha < 1$ and where $\overline{\mathcal{D}}$ denotes the closure of $\mathcal{D}$.

The function $\mathcal{L}F$ referred to in (4.63) takes the form

$$(\mathcal{L}F)(x) = \frac{x-b}{a-b}F(a) + \frac{x-a}{b-a}F(b) \quad . \tag{4.66}$$

The function $E = F - \mathcal{L}F$ then satisfies $\phi' E \in B(\mathcal{D})$ [36] , and moreover, for all $z \in \overline{\mathcal{D}}$,

$$|E(z)| \leq C|(z-a)(z-b)|^{\alpha} \tag{4.67}$$

where C is a constant. In particular, the integrals

$$(\mathcal{H}E)(a) = \int_\Gamma \frac{E(t)}{t-a}\,dt \ , \ (\mathcal{H}E)(b) = \int_\Gamma \frac{E(t)}{t-b}\,dt \tag{4.68}$$

exist and are finite. Let p and q be polynomials defined by

$$\begin{aligned} p(t) &= \frac{6}{(b-a)^3}(t-a)(t-b)(t-\tfrac{2}{3}b-\tfrac{1}{3}a) \\ q(t) &= \frac{6}{(b-a)^3}(t-a)(t-b)(t-\tfrac{2}{3}a-\tfrac{1}{3}b) \quad . \end{aligned} \tag{4.69}$$

Then, define P and Q by

$$\begin{aligned} P(x) = (\mathcal{H}p)(x) &= p(x)\log(\tfrac{b-x}{x-a}) + (b-a)p'(x) \\ &+ \tfrac{1}{4}\{(b-x)^2-(a-x)^2\}p''(x) \\ &+ \tfrac{1}{18}\{(b-x)^3-(a-x)^3\}p'''(x) \\ Q(x) = (\mathcal{H}q)(x) &= q(x)\log(\tfrac{b-x}{x-a}) + (b-a)q'(x) \\ &+ \tfrac{1}{4}\{(b-x)^2-(a-x)^2\}q''(x) \\ &+ \tfrac{1}{18}\{(b-x)^3-(a-x)^3\}q'''(x) \end{aligned} \tag{4.70}$$

The function p, q, P and Q have the following properties:

$$p(a) = q(a) = p(b) = q(b) = 0$$

$$(4.71) \qquad P(a) = Q(b) = 1$$

$$P(b) = Q(a) = 0$$

Now let us define G by

$$(4.72) \qquad G(t) = E(t)-p(t)(\mathcal{H}E)(a)-q(t)\mathcal{H}E(b)$$

The function G has the following properties:

$$\phi'G \in B(\mathcal{D}) \quad ;$$

$$G \in \mathrm{Lip}_\alpha(\overline{\mathcal{D}}) \quad ;$$

$$(4.73) \qquad G \quad \text{satisfies (4.67) in } \overline{\mathcal{D}} \quad ;$$

$$(\mathcal{H}G)(x) \quad \text{exists for all} \quad x \in \Gamma$$

$$(\mathcal{H}G)(a) = \mathcal{H}G(b) = 0 \quad ;$$

$$N^*(G,\mathcal{D}) = \sup_{x\in\Gamma} N(\frac{G}{\cdot-x}, \mathcal{D}) < \infty \quad .$$

Upon replacing t by $\psi(u)$, we get

$$(4.74) \qquad \text{P.V.} \int_\Gamma \frac{G(t)}{t-x}\,dt = \text{P.V.} \int_{\mathbb{R}} \left\{G(\psi(u))\ \frac{u-\phi(x)\psi'(u)}{\psi(u)-x}\right\} \cdot \frac{du}{u-\phi(x)}$$

Now using (3.40) on this last integral yields

$$\left| \text{P.V.} \int_\Gamma \frac{G(t)}{t-x}\,dt + 2h \sum_{k=-\infty}^{\infty} \frac{G(z_k)}{\phi'(z_k)} \frac{\sin^2\{\frac{\pi}{2h}[\phi(x)-kh]\}}{x-z_k} \right|$$

(4.75)

$$\leqq \frac{1}{2} \frac{N^*(G,\mathcal{D})}{\sinh(\pi d/h)} [1+e^{-\pi d/h}]$$

Collecting the above results and using the (3.49) approximate form of (3.47), we get the following theorem.

THEOREM 4.7: Let $F \in B(\mathcal{D})$, where $\mathcal{D}$ is bounded, and let $F \in \mathrm{Lip}_\alpha(\bar{\mathcal{D}})$, where $0 < \alpha < 1$. Let $h = [\pi d/(\alpha N)]^{1/2}$. Then for all $x \in \Gamma$

$$\left| \text{P.V.} \int_\Gamma \frac{F(t)}{t-x}\,dt - [F(b)-F(a)] - (\mathcal{L}F)(x)\log\left(\frac{b-x}{x-u}\right) \right.$$

$$-P(x)(\mathcal{H}E)(a) - Q(x)(\mathcal{H}E)(b)$$

(4.76)
$$\left. + h \sum_{k=-N}^{N} \left\{ \sum_{\substack{j=-N \\ j\neq k}}^{N} \frac{G(z_j)}{\phi'(z_j)} \frac{[1-(-1)^{k-j}]}{z_k-z_j} S(k,h)\circ\phi(x) \right| \right.$$

$$\leqq C_1 N^{3/2} e^{-(\pi d\alpha N)^{1/2}}$$

where $E = F-\mathcal{L}F$, $\mathcal{L}F$ is defined in Eq. (4.66), G in (4.72), p and q in (4.69), P and Q in (4.70) and where C_1 depends only on F , d and α .

EXAMPLE 4.14: The Hilbert Transform over $[0,\infty]$. Let us make the following assumptions:

(i) $F \in B(\mathcal{D})$, where $\mathcal{D}$ is defined in (4.11);

(ii) $F \in \mathrm{Lip}_\alpha \{z : |\arg z| \leqq d \ , \ 0 \leqq |z| \leqq \rho\}$, for some $\rho > 0$

Let E and G be defined by

$$E(t) = F(t) - F(0)/(1+t)$$

(4.77)

$$G(t) = E(t) - (\mathcal{H}E)(0)\; t\,/(1+t)^2$$

The formula analogous to (4.72) then becomes

$$\left| \text{P.V.} \int_0^\infty \frac{G(t)}{t-x}\, dt + 2h \sum_{k=-\infty}^{\infty} e^{kh} G(e^{kh}) \frac{\sin^2\{\frac{\pi}{2h}[\log x - kh]\}}{x - e^{kh}} \right|$$

(4.78)

$$\leq \frac{1}{2} \frac{N^*(G,\mathcal{D})}{\sinh(\frac{\pi d}{h})} [1 + e^{-\pi d/h}]$$

for all $x \in [0,\infty]$ and for all $h > 0$, where

$$(4.79)\quad N^*(G,\mathcal{D}) = \sup_{x \in (0,\infty)} \left\{ \int_0^\infty \left| \left| \frac{G(\rho e^{-id^-})}{\rho e^{id} - x} \right| + \left| \frac{G(\rho e^{-id^-})}{\rho e^{-id} - x} \right| \right| d\rho \right\}$$

Thus be choosing $h = [\pi d/(\alpha N)]^{1/2}$, it follows that

$$\left| \text{P.V.} \int_0^\infty \frac{F(t)}{t-x}\, dt - \frac{F(0)}{1+x} \log(1/x) \right.$$

$$- \left[\frac{x}{(1+x)^2} \log(1/x) + \frac{1}{1+x}\right] (\mathcal{H}E)(0)$$

(4.80)

$$\left. + h \sum_{k=-N}^{N} \left\{ \sum_{\substack{j=-N \\ j \neq k}}^{N} e^{jh} G(e^{jh}) \frac{[1-(-1)^{k-j}]}{e^{kh} - e^{jh}} \right\} \cdot S(k,h) \circ \log x \right|$$

$$\leq C_1\, N^{3/2} e^{-(\pi d \alpha N)^{1/2}}$$

for all $x \in [0,\infty]$, where C_1 depends only on F, d and α .

We remark that in applications, the integrals $(\mathcal{H}E)(a)$ and $(\mathcal{H}E)(b)$ in (4.68) are evaluated by means of formula (4.32) and the integral $(\mathcal{H}E)(0) = \int_0^\infty [E(t)/t]dt$ in (4.80) is evaluated by means of formula (4.36).

We remark also that (4.76) and (4.80) provide a convenient expression for solving Hilbert-type problems (see e.g. [11]) and integral equations with Cauchy-type singularities over closed contours.

EXAMPLE 4.15: The Integral $\mathcal{U}_{\beta,y}F$ for $\Gamma = [0,1]$. In this case we want to approximate

$$(\mathcal{U}_{\beta,y}f)(x) = \int_0^1 [(x-t)^2+y^2]^{-\beta-\frac{1}{2}} f(t)dt \tag{4.81}$$

where $f \in B(\mathcal{D})$, $\mathcal{D}$ as in (4.7), and where $F \in \mathrm{Lip}_\alpha(\bar{\mathcal{D}})$, with $0 < \alpha < 1$. Using the notation $F(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k\, k!} x^k$ for the hypergeometric function, it may be shown that if n is a nonnegative integer, then

$$\begin{aligned} w_n(x,y) &\equiv \int_0^1 t^n[(x-t)^2+y^2]^{-\beta-\frac{1}{2}}\, dt \\ &= x^{n+1}(x^2+y^2)^{-\beta-\frac{1}{2}} F(\beta+\tfrac{1}{2},1;\tfrac{n+3}{2};\tfrac{x^2}{x^2+y^2}) \\ &\quad + (1-x)^{n+1}[(1-x)^2+y^2]^{-\beta-\frac{1}{2}} F(\beta+\tfrac{1}{2},1;\tfrac{n+3}{2};\tfrac{(1-x)^2}{(1-x)^2+y^2}) \end{aligned} \tag{4.82}$$

Let us define $\mathcal{L}f$, E and G by

$$\begin{aligned} (\mathcal{L}f)(x) &= (1-x)f(0) + xf(1) \\ E &= F - \mathcal{L}F \\ G(x) &= E(x) - x(1-x)\{(ax+b)(\mathcal{U}_{\beta,y}f)(0) + (cx+d)(\mathcal{U}_{\beta,y}f)(1)\} \end{aligned} \tag{4.83}$$

where

$$a = (3w_2 - 2w_2 - 2w_3 - w_1)^{-1} \quad , \; c = -a$$

(4.84) $$b = (w_3 - 2w_2 + w_1)(w_2 - w_1)^{-1}a \quad , \quad d = -(w_2 - w_3)(w_1 - w_2)^{-1}a \quad ,$$

$$w_n = w_n(0,y)$$

By taking $h = [\pi d/(\alpha N)]^{1/2}$, we thus find by proceeding on for (3.51), that

$$\left| \int_0^1 [(x-t)^2 + y^2]^{-\beta-\frac{1}{2}} f(t)dt - w_0(x,y)f(0) \right.$$

$$- w_1(x,y)[f(1)-f(0)]$$

$$- [w_2(x,y)-w_3(x,y)][a(\mathcal{U}_{\beta,y}E)(0)+c(\mathcal{U}_{\beta,y}E)(1)]$$

(4.85) $$- [w_1(x,y)-w_2(x,y)][b(\mathcal{U}_{\beta,y}E)(0)+d(\mathcal{U}_{\beta,y}E)(1)]$$

$$- \sum_{k=-N}^{N} \left\{ \sum_{j=-N}^{N} \frac{G(z_j)}{\phi'(z_j)} \left| \frac{(z_k - z_j)^2 + y^2}{(kh-jh)^2 + \phi(y)^2} \right|^{-\beta-\frac{1}{2}} \tau_{j-k}(\phi(y), \beta, h) \right\} \cdot$$

$$\left. \cdot S(k,h) \circ \phi(x) \right| \leqq C_1 \, N^{3/2} e^{-(\pi d \alpha N)^{1/2}}$$

for all $x \in [0,1]$, where $\phi(x) = \log[x/(1-x)]$, τ_k is defined in (2.29) , z_k in (4.8) and where C_1 depends only on f, d and α . In applications $(\mathcal{U}_{\beta,y}E)(0)$ and $(\mathcal{U}_{\beta,y}E)(1)$ are evaluated by means of formula (4.32). It may also be convenient to approximate $w_n(x,y)$ by $[\mathcal{L}w_n(\cdot,y)](x)$ $+ \sum_{k=-N}^{N} \{w_n(z_k,y) - [\mathcal{L}w_n(\cdot,y)](z_k)\} S(k,h) \circ \log \frac{x}{1-x}$, where $z_k = \frac{1}{2} + \frac{1}{2}\tanh(kh/2)$ and where $[\mathcal{L}w_n(\cdot,y)](x) = (1-x)w_n(0,y) + xw_n(1,y)$.

EXAMPLE 4.16: The Integral $(V_\beta f)$ for $\Gamma = [0,1]$. Let f satisfy the condition of the previous example. We want to approximate

$$(V_\beta f)(x) = \int_0^1 |x-t|^{\beta-1} f(t)dt \ , \ \beta > 0 \ . \tag{4.86}$$

As above,

$$\begin{aligned} (Lf)(x) &= (1-x)f(0)+xf(1) \\ E &= f - Lf \\ G(x) &= E(x) - x(1-x)[(a+bx)(V_\beta E)(0)+(c+dx)(V_\beta E)(1)] \end{aligned} \tag{4.87}$$

$$\begin{aligned} a &= \frac{(\beta+2)(\beta+3)}{\beta-1} & c &= -\frac{(\beta+1)(\beta+2)(\beta+3)}{\beta-1} \\ b &= \frac{-2(\beta+2)}{\beta-1} & d &= \frac{(\beta+1)^2\beta+2}{\beta-1} \end{aligned} \tag{4.88}$$

Then, by setting

$$\begin{aligned} I_n(x) &= \int_0^1 t^n |t-x|^{\beta-1} dt \\ &= \frac{n!}{(\beta+1)_n} x^{\beta+n} + (1-x)^\beta \sum_{k=0}^{n} \binom{n}{k} \frac{x^{n-k}(1-x)^k}{\beta+k} \end{aligned} \tag{4.89}$$

for $n = 0,1,2,3,$ and taking $h = [\pi d/(\alpha N)]^{1/2}$, we have

$$\Bigg| \int_0^1 |x-t|^{\beta-1} f(t)dt - [I_0(x)f(0)+\{I_1(x)-I_0(x)\}f(1)]$$

$$+ [a\{I_3(x)-I_2(x)\}+b\{I_2(x)-I_1(x)\}] \int_0^1 t^{\beta-1}E(t)dt$$

$$+ [c\{I_3(x)-I_2(x)\}+d\{I_2(x)-I_1(x)\}] \int_0^1 (1-t)^{\beta-1}E(t)dt$$

(4.90)
$$- h \frac{\Gamma(\beta)\cos(\frac{\pi\beta}{2})}{\pi} \sum_{k=-N}^{N} \Bigg\{ \frac{h^{\beta-1}\pi^{1-\beta}}{1-\beta} \frac{G(z_k)}{\phi'(z_k)^\beta}$$

$$+ \sum_{\substack{j=-N \\ j\neq k}}^{N} \frac{G(z_j)}{\phi'(z_j)} \left| \frac{z_j - z_k}{j-k} \right|^{\beta-1} \int_0^\pi t^{-\beta} \cos[(k-j)t]dt \Bigg\} S(k,h)\circ\phi(x) \Bigg|$$

$$\leq C_1 \, N^{3/2} e^{-(\pi d\alpha N)^{1/2}}$$

for all $x \in [0,1]$, where z_j and $\phi(x)$ are defined in (4.8), and where C_1 is a constant depending only on F, d, α and β.

<u>EXAMPLE 4.17</u>: <u>The Integral $V_\beta f$ for $\Gamma = [0,\infty]$.</u> Let f satisfy the condition in Ex.4.14. We shall approximate the integral

(4.91)
$$\int_0^\infty |x-t|^{\beta-1} f(t)dt \quad , \quad 0 < \beta < 1 .$$

To this end, we set

$$(Lf)(x) = \frac{f(0)}{1+x}$$

(4.92)
$$E = f - Lf$$

$$G(x) = E(x) - a \frac{x}{(1+x)^2} (V_\beta E)(0)$$

where

$$a = \frac{\sin(\pi\beta)}{\pi\beta} \tag{4.93}$$

Then, setting

$$\begin{aligned} I_0(x) &= \int_0^\infty |x-t|^{\beta-1} \frac{dt}{1+t} \\ &= \frac{x^\beta}{\beta(1+x)} F(1,\beta;1+\beta;\frac{x}{1+x}) + \frac{\pi}{\sin(\pi\beta)} (1+x)^{\beta-1} \end{aligned}$$

$$\begin{aligned} I_1(x) &= \int_0^\infty |x-t|^{\beta-1} \frac{tdt}{(1+t)^2} = \\ &= \frac{1}{\beta(\beta+1)} \frac{x^{\beta+1}}{(1+x)^2} F(2,\beta;2+\beta;\frac{x}{(1+x)^2}) + \frac{\pi(x+\beta)}{\sin(\pi\beta)(1+x)^{2+\beta}} \end{aligned} \tag{4.94}$$

and taking $h = [\pi d/(\alpha' N)^{1/2}]$ where $\alpha' = \min(\alpha,\beta)$, we have

$$\begin{aligned} \Bigg| \int_0^\infty &|x-t|^{\beta-1} f(t)dt - f(0)I_0(x) - a(V_\beta E)(0)I_1(x) \\ &- \frac{h\Gamma(\beta)\cos(\frac{\pi\beta}{2})}{\pi} \sum_{k=-N}^{N} \Bigg\{ \frac{(\pi/h)^{1-\beta}}{1-\beta} e^{\beta kh} G(e^{kh}) \\ &+ \sum_{\substack{j=-N \\ j\neq k}}^{N} e^{jh} G(e^{jh}) \left| \frac{e^{jh}-e^{kh}}{j-k} \right|^{\beta-1} \int_0^\pi t^{-\beta} \cos[(k-j)t]dt \Bigg\} \cdot \\ &\cdot S(k,h) \circ \log x \Bigg| \leq C_1 N^{3/2} e^{-(\pi d\alpha' N)^{1/2}} \end{aligned} \tag{4.95}$$

for all $x \in [0,\infty]$, where C_1 depends only on f, β, d and α' .

In applications $(V_\beta E)(0)$ is evaluated by means of formula (4.36). It may also be convenient to approximate the terms in $I_0(x)$ and $I_1(x)$ involving hypergeometric functions by (4.26). The 2N integrals

$\int_0^{\pi} t^{-\beta}\cos(kt)dt = \pi^{1-\beta} \int_0^1 t^{-\beta} \cos(\pi kt)dt$, $k=1,2,\ldots 2N$ may be evaluated using (4.32).

<u>EXAMPLE 4.18</u>: <u>The Integral $(\mathcal{W}F)$ for $\Gamma = [0,1]$.</u> Let F satisfy the condition in Ex. 4.15. We want to approximate

$$(\mathcal{W}F)(x) = \int_0^1 \log|x-t|F(t)dt \quad . \tag{4.96}$$

We again set

$$\begin{aligned} (\mathcal{L}F)(x) &= (1-x)F(0)+xF(1) \\ E &= F - \mathcal{L}F \\ G(x) &= E(x)-(1-x)[(a+bx)(\mathcal{W}E)(0)+(c+dx)\mathcal{W}E(1)] \end{aligned} \tag{4.97}$$

where

$$\begin{aligned} a &= -24 \qquad & c &= 24 \\ b &= \frac{42}{5} \qquad & d &= -\frac{78}{5} \end{aligned} \tag{4.98}$$

Setting

$$\begin{aligned} I_n(x) &= \int_0^1 \log|x-t|\,t^n dt \\ &= \frac{1}{n+1}\left\{x^{n+1}\log x + (1-x)^{n+1}\log(1-x) \right. \\ &\qquad \left. - \sum_{k=0}^{n} \frac{x^k}{n+1-k}\right\} \end{aligned} \tag{4.99}$$

for $n = 0,1,2,3,$ and taking $h = [\pi d/(\alpha N)]^{1/2}$, we have

$$
\begin{aligned}
&\Big| \int_0^1 \log|x-t|F(t)dt - [I_0(x)f(0)+\{I_1(x)-I_0(x)\}f(1)] \\
&+ [a\{I_3(x)-I_2(x)\}+b\{I_2(x)-I_1(x)\}] \int_0^1 E(t)\log t\, dt \\
&+ [c\{I_3(x)-I_2(x)\}+d\{I_2(x)-I_1(x)\}] \int_0^1 E(t)\log(1-t)dt \\
&- h \sum_{k=-N}^{N} \left\{ \frac{G(z_k)}{\phi'(z_k)} \log \frac{1}{\phi'(z_k)} \right. \\
&\left. + \sum_{\substack{j=-N \\ j\neq k}}^{N} \frac{G(z_j)}{\phi'(z_j)} \left[\log\left|\frac{z_k-z_j}{kh-jh}\right| + \frac{1}{2\pi}\int_0^{\pi} \frac{1-\cos[(k-j)t]}{t} dt \right] \right\} S(k,h)\circ\phi(x) \Big| \\
&\leqq C_1\, N^{3/2} e^{-(\pi d\alpha N)^{1/2}}
\end{aligned} \tag{4.100}
$$

where $\phi(x) = \log[x/(1-x)]$, and where C_1 depends only on f,d and α .

In applications, the integrals $\int_0^1 E(t)\log t\, dt$, $\int_0^1 E(t)\log(1-t)dt$ and $\int_0^{\pi} \frac{1-\cos kt}{t} dt$ $(k=1,2,\ldots,2N)$ are evaluated using (4.32).

<u>EXAMPLE 4.19:</u> <u>The Integral WF for $\Gamma = [0,\infty]$.</u> In this case we shall give an approximate expression for the integral

$$(WF)(x) = \int_0^{\infty} \log|x-t|F(t)dt \tag{4.101}$$

where F satisfies the conditions in Ex. 4.14. We assume also that $(WF)(x)$ exists for all $x \in [0,\infty)$. Proceeding as in the previous examples, we set

$$(\mathcal{L}F)(x) = \frac{F(0)}{(1+t)^2}$$

$$E = F - \mathcal{L}F$$

(4.102)

$$G(x) = E(x) - \frac{x}{(1+x)^3}\left(a + \frac{b}{1+x}\right)$$

$$a = \tfrac{1}{4}\int_0^\infty [4\log t + 6]E(t)dt$$

$$b = \tfrac{1}{4}\int_0^\infty [18 - \log t]E(t)dt$$

Let us also set

$$I_0(x) = \int_0^\infty \frac{\log|x-t|}{(1+t)^2}\,dt = \frac{x}{1+x}\log x$$

$$I_1(x) = \int_0^\infty \frac{t\log|x-t|}{(1+t)^3}\,dt$$

(4.103)

$$= \frac{\tfrac{1}{2}x^2}{(1+x)^2}\log x + \tfrac{1}{2}\frac{1}{1+x}\left\{1 + \frac{1}{(1+x)(2+x)}\right\}$$

$$I_2(x) = \int_0^\infty \frac{t\log|x-t|}{(1+t)^4}\,dt$$

$$= \frac{1}{6}\left\{1 - \frac{1}{(1+x)^2}\right\}\log x + \frac{1}{3}\frac{x}{(1+x)^3}\log\frac{2+x}{1+x}$$

$$+ \frac{1}{6}\frac{4x+6}{(1+x)^2(2+x)^2} - \tfrac{1}{2}\frac{1}{1+x} + \frac{1}{6}\frac{(x+2)^2-4}{(1+x)^3}$$

Then, by taking $h = [\pi d/(\alpha N)]^{1/2}$, we get

$$\Bigg| \int_0^\infty \log|x-t|F(t)dt - I_0(x)F(0) - aI_1(x) - bI_2(x) - h\sum_{k=-N}^{N} \Bigg\{kh\, e^{kh}G(e^{kh}) + \sum_{\substack{j=-N\\ j\neq k}}^{N} e^{jh}G(e^{jh})\Bigg[\log\left|\frac{e^{kh}-e^{jh}}{kh-jh}\right| + \frac{1}{2\pi}\int_0^\pi \frac{1-\cos[(k-j)t]}{t}\,dt\Bigg]\Bigg\} S(k,h)\circ\log x\Bigg| \leq C_1\, N^{3/2} e^{-(\pi d\alpha N)^{1/2}} \tag{4.104}$$

for all $x \in [0,\infty]$, where C_1 depends only on F, d and α .

5. APPROXIMATION OF TRANSFORMS ON Γ .

The approximation of Fourier transforms over $(-\infty,\infty)$ was discussed in Sec. 3, where the trapezoidal rule yields the FFT method. In this section we briefly describe methods for approximating the semi-infinite Fourier, the Laplace, the Mellin and the Hankel transforms, namely

$$(5.1) \qquad \mathcal{F}(F,\lambda) = \int_0^\infty F(t)e^{i\lambda t}dt$$

$$(5.2) \qquad \mathcal{L}(F,\lambda) = \int_0^\infty F(t)e^{-\lambda t}dt$$

$$(5.3) \qquad \mathcal{M}(F,\lambda) = \int_0^\infty F(t)t^{\lambda-1}dt$$

and

$$(5.4) \qquad \mathcal{H}_\nu(F,\lambda) = \int_0^\infty F(t)J_\nu(\lambda t)dt$$

respectively [24].

The integral $\mathcal{F}(F,\lambda)$ may arise as a sine or cosine transform, in the process of approximating $\int_{\mathbb{R}} F(t)e^{ixt}\,dt$, when F has one or more integrable singularities on $\mathbb{R}$. More generally, in the approximation of any of the above integrals, if F has a finite number of isolated singularities on the interval of integration, then we recommend splitting up each integral into a finite number of integrals, in such a way that singularities occur only at the end-points of each interval. Over any such finite interval (a,b) , we recommend using the formula

$$\int_a^b G(t)dt \cong (b-a)h \sum_{k=-N}^{N} \frac{e^{kh}}{(1+e^{kh})^2} G\left(\frac{a+be^{kh}}{1+e^{kh}}\right) \tag{5.5}$$

while over $(0,\infty)$ we recommend using one of the formulas (4.36) or (4.38). Usually (4.36) works well for (5.2) and (5.3), while (4.38) works well for (5.1) and (5.4).

An additional difficulty occurs for the application of (4.38) to the approximation of (5.1) and (5.4), where the integrand decreases slowly. Consider for example, the approximation of

$$\mathcal{F}(F,\lambda) = \int_0^\infty t^{1/2} \frac{\sin \lambda t \, dt}{1+t} \tag{5.6}$$

by a truncation of

$$S(F,\lambda) = h \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{1+e^{-2kh}}} \frac{z_k^{1/2}}{1+z_k^{1/2}} \sin(\lambda z_h) \tag{5.7}$$

where $z_k = \log[e^{kh} + \sqrt{1+e^{2kh}}]$. This series sum (5.7) is quite accurate if $h < \min(1, \pi/\lambda)$. For large positive k , $z_k \sim kh + \log 2$; hence if $\lambda > 1$ and $h < \pi/\lambda$ at least one of the points z_k falls between every consecutive pair of zeros of $\sin \lambda t$, and $S(F,\lambda)$ is then a very accurate approximation of $\mathcal{F}(F,\lambda)$. We then recommend splitting the series $S(F,\lambda)$ into two parts :

$$h \sum_{k=-\infty}^{\infty} = h \sum_{k=-\infty}^{N} + h \sum_{k=N+1}^{\infty} \quad .$$

The integer N is chosen so that $z_k^{1/2}/(1+z_h) < 1/2$ (say) if $k > N$, and so that the series $\sum_{k=-\infty}^{N}$ includes all and only the terms for which

$z_k < \mu_\circ \pi/\lambda$, where $\mu_\circ$ is a positive integer. Direct summation and truncation of the infinite series $h \sum_{k=-\infty}^{N}$ produces no difficulty, since this tail of the series $h \sum_{k=-\infty}^{\infty}$ converges very rapidly. The series $\sum_{k=N+1}^{\infty}$ is summed by evaluating the first few

$$(5.8) \qquad t_\mu = h \sum_{\substack{\frac{\mu\pi}{\lambda} < z_k < \frac{(\mu+1)\pi}{\lambda}, \\ \mu \geq \mu_\circ}} \frac{1}{\sqrt{1+e^{-2kh}}} \frac{z_k^{1/2}}{1+z_k} \sin(\lambda z_k)$$

and then applying Euler's method of summation to approximate the alternating series $\sum_{\mu \geq \mu_\circ} t_\mu$.

The zeros of $J_\nu(\lambda t)$ are asymptotically equi-spaced for large t and the above described procedure may be similarly applied to the approximation of $H_\nu(F,\lambda)$.

The approximation of each of the four integrals (5.1)-(5.4) by the above outlined procedure is described in detail in [24], where many examples are considered, illustrating the accuracy and superiority of these methods over other methods. It is furthermore shown in [24] that these methods may be used effectively for λ up to 100. If $\lambda > 100$, we recommend asymptotic method [3,31,41,42,59].

6. APPROXIMATE SOLUTION OF DIFFERENTIAL EQUATIONS VIA THE SINC-GALERKIN METHOD.

In this section we shall illustrate the application of the cardinal function approximations to the solution of second order boundary value problems [53]. We defer the solution of initial value problems to Sec. 7, where we illustrate their solution as a special case of the solution of Volterra integral equations.

The method of approximate solution of ordinary and partial differential equation boundary value problems which we present here is carried out by the Galerkin scheme [44]. It is perhaps best illustrated by considering the solution of the simple second order linear boundary value problem

$$(\mathcal{L}f)(x) = f''(x) + \mu(x)f'(x) + \nu(x)f(x) - \tau(x) = 0, \quad x \in \Gamma \tag{6.1}$$

$$f(a) = f(b) = 0 \,. \tag{6.2}$$

Other illustrations of solutions of ordinary and partial differential equation boundary value problems are given in examples at the end of this section. Throughout this section we shall consider only the solution of second order boundary value problems.

Let $A(u)$ denote a diagonal matrix with diagonal elements $(u_{-N}, u_{-N+1}, \ldots u_N)$, where $u_k = u(z_k)$, let $\underline{1}$ denote the vector $(1,1,\ldots,1)^T$, and let $I^{(1)}$ and $I^{(2)}$ denote the matrices

$$(6.3)\qquad I^{(1)} = [\delta_{ij}^{(1)}] = \begin{bmatrix} 0 & -1 & \frac{1}{2} & -\frac{1}{3} & \cdots & \frac{1}{2N} \\ 1 & 0 & -1 & \frac{1}{2} & \cdots & -\frac{1}{2N-1} \\ & & & \cdots & & \\ -\frac{1}{2N} & \frac{1}{2N-1} & -\frac{1}{2N-2} & \cdots & & 0 \end{bmatrix}$$

$$(6.4)\qquad I^{(2)} = [\delta_{ij}^{(2)}] = \begin{bmatrix} -\frac{\pi^2}{3} & \frac{2}{1^2} & -\frac{2}{2^2} & \frac{2}{3^2} & \cdots & -\frac{2}{(2N)^2} \\ \frac{2}{1^2} & -\frac{\pi^2}{3} & \frac{2}{1^2} & -\frac{2}{2^2} & \cdots & \frac{2}{(2N-1)^2} \\ & & & \cdots & & \\ -\frac{2}{(2N)^2} & \frac{2}{(2N-1)^2} & -\frac{2}{(2N-2)^2} & \frac{2}{(2N-3)^2} & \cdots & -\frac{\pi^2}{3} \end{bmatrix}$$

Let us now make some assumptions on $\mathcal{L}f$ in (6.1). Using the notation of Definition 4.1, let μ,ν and σ be analytic in $\mathcal{D}$, such that (6.1)-(6.2) has a unique* solution f for which

$$(6.5)\qquad f''/\phi' \quad , \quad f'\mu/\phi' \quad , \quad f\nu/\phi' \quad \text{and} \quad \sigma/\phi' \in B(\mathcal{D}) .$$

and such that f satisfies (4.15) on Γ.

We approximate f on Γ by

$$(6.6)\qquad f(x) \cong f_N(x) \equiv \sum_{k=-N}^{N} f_k\, S(k,h)\circ\phi(x) \quad .$$

---

*The assumption of uniqueness may be bypassed via the use of the generalized inverse.

The Galerkin scheme enables us to determine $f_k \cong f(z_k)$ by solving the linear system of equations

$$(6.7) \qquad (\mathcal{L}f_N, S(k,h)\circ\phi) = 0 \ , \ k=-N,-n+1,\ldots,N \ ,$$

where the inner product in (6.7) is (nearly always) defined (for second order problems) by

$$(6.8) \qquad (u,v) = \int_\Gamma \frac{1}{\phi'(x)} u(x)(\overline{v(x)})\,dx$$

THEOREM 6.1: Let the above assumptions be satisfied. There exist constants $C_1,\ldots,C_6$, depending only on f, d and α , such that if $h = [\pi d/(\alpha N)]^{1/2}$, then

$$(6.9) \qquad \left| \int_\Gamma \frac{\sigma(x)}{\phi'(x)} S(k,h)\circ\phi(x)dx - h\frac{\sigma_k}{\phi_k'^2} \right| \leq C_1 N^{-1/2} e^{-(\pi d\alpha N)^{1/2}}$$

$$(6.10) \qquad \left| \int_\Gamma \frac{\nu(x)f(x)}{\phi'(x)} S(k,h)\circ\phi(x)dx - \frac{h\,\nu_k f(z_k)}{\phi_k'^2} \right| \leq C_2 N^{-1/2} e^{-(\pi d\alpha N)^{1/2}}$$

$$(6.11) \qquad \left| \int_\Gamma \frac{\mu(x)f'(x)}{\phi'(x)} S(k,h)\circ\phi(x)dx - \frac{h\,\mu_k f'(z_k)}{\phi_k'^2} \right| \leq C_3 N^{-1/2} e^{-(\pi d\alpha N)^{1/2}}$$

$$(6.12) \qquad \left| \int_\Gamma \frac{\mu(x)f'(x)}{\phi'(x)} S(k,h)\circ\phi(x)dx - \frac{\mu_k}{\phi_k'} \sum_{j=-N}^{N} f(z_j)\delta_{jk}^{(1)} \right| \leq C_4 e^{-(\pi d\alpha N)^{1/2}}$$

$$(6.13) \qquad \left| \int_\Gamma \frac{f''(x)}{\phi'(x)} S(k,h)\circ\phi(x)dx - \frac{h\,f''(z_k)}{\phi_k'^2} \right| \leq C_5 N^{-1/2} e^{-(\pi d\alpha N)^{1/2}}$$

$$(6.14) \qquad \left| \int_\Gamma \frac{f''(x)}{\phi'(x)} S(k,h)\circ\phi(x)dx - h\sum_{j=-N}^{N} f(z_j)\left\{\frac{\phi_k''}{\phi_k'^2}\,\delta_{jk}^{(1)} + \frac{1}{h}\,\delta_{jk}^{(2)}\right\} \right|$$

$$\leq C_6 N^{1/2} e^{-(\pi d\alpha N)^{1/2}}$$

Let $f_n = (f_{-N}, f_{-N+1}, \ldots, f_N)^T$ be the solution of the system

$$[\frac{1}{h} I^{(2)} - A(\frac{\phi''+\mu\phi'}{\phi'^2}) I^{(1)} + hA(\frac{\nu}{\phi'^2})] f_n = hA(\frac{\sigma}{\phi'^2}) \underline{1} \tag{6.15}$$

which is obtained from (6.7). Then the function f_N defined in (6.6) then satisfies

$$|f(x) - f_N(x)| \leqq C_1 \, N^{3/2} e^{-(\pi d \alpha N)^{1/2}} \tag{6.16}$$

for all $x \in \Gamma$, where C_1 depends only on f, d and α.

We remark that the approximations (6.9), (6.10), (6.11), and (6.13) may be obtained by applying the formula (4.31) to the respective integrals. The approximations (6.12) and (6.14) may be obtained by replacing f by f_N and then applying (4.31). A different family of approximations is also possible, such as that obtained in [53]. For example, the above formulas yield the approximation

$$I = \int_\Gamma f(x) \, [\frac{d}{dx} \sum_j c_j \; S(j,h) \circ \phi(x)] S(k,h) \circ \phi(x) dx$$

$$\cong f(x_k) \sum_j c_j \; \delta_{jk}^{(1)}$$

On the other hand, if e.g. f is bounded on Γ we find, after integration by parts, and then applying the above approximations, that

$$I = - \int_\Gamma \; [\sum_j c_j S(j,h) \circ \phi(x)] \; \frac{d}{dx} \; [f(x) S(k,h) \circ \phi(x)] dx$$

$$\cong \sum_j c_j \left\{ \frac{h}{\phi_j'} f'(x_j) \delta_{jk}^{(0)} + f(x_j) \delta_{kj}^{(1)} \right\}$$

Both approximations have the same order of accuracy. The approximations of Theorem 6.1 are usually simpler in form than those we obtain after integration by parts.

We also remark that the explicit expressions (6.9) - (6.14) make "collocation" and "Galerkin" synonymous for this method.

The matrix $I^{(2)}$ is the dominant matrix of the system (6.15). It is a symmetric, negative definite matrix, with eigenvalues $-\lambda_k, k=-N,-N+1,\ldots,N$, where $\pi^2/(N+1)^2 < \lambda_k < \pi^2$. Thus $I^{(2)}$ is a well-conditioned matrix, with condition number less than $(N+1)^2$. The matrix $I^{(1)}$ is a skew-symmetric matrix with determinant zero. It has eigenvalues iw_k, where $-\pi < w_k < \pi$. Contrary to the case of finite difference or finite element methods which lead to sparse matrices, the matrix in (6.15) is a full matrix. However, the rate of convergence (6.16) of the above method is considerably faster than that of finite difference or finite element methods, which converge at the rate $O(n^{-q})$ for a system of order n, where for one-dimensional problems, q is usually 1 or 2. Moreover, under the above assumptions, the $O(e^{-cn^{\frac{1}{2}}})$ rate of convergence cannot be improved (see Sec. 9), regardless of the basis. Due to its rapid convergence, the present method yields a desired accuracy with a relatively small system of equations.

The reduction in the amount of work required is considerably greater in two and more dimensions. In application, the coefficients of differential equations in p dimensions are piecewise analytic functions in each variable. Singularities of the solutions occur wherever the coefficients of the equations have singularities but this occurs only on $p-1$ dimensional surfaces. Thus (with the exception of inverse problems, where determination of the boundaries is more difficult) we can determine a priori the points, or surfaces where the singularities occur, and using a system of order n, we can achieve as approximation having an $O(\exp[-cn^{1/(2p)}])$ error. This should be compared with methods based on finite differences, or on polynomial, or finite element-type approximations, for which the error is $O(n^{-c/p})$ in p dimensions.

We do not have a precise idea at this time, by how much faster we can solve partial differential equations by the above method than by finite difference or finite element methods. The solutions of two-dimensional "model" problems in the examples which follow seem to indicate that we can get by with less than 1/3 of the work required of classical methods to achieve 3 places of accuracy, and less than 1/10 of the work it takes for 5 places. Preliminary calculations indicate the reduction in the amount of work required to solve 3 and higher dimensional problems is considerably greater, e.g. by a factor of 100 in 3 dimensions.

The above method also easily reduces a nonlinear equation to an algebraic system. For example, for the case of the problem

$$\begin{aligned} L(f) &= f''(x) + G(x,f(x),f'(x)) = 0 \\ f(a) &= f(b) = 0 \quad , \end{aligned} \tag{6.17}$$

if $G(\cdot,f,f')/\phi' \in B(\mathcal{D})$, then we can make the approximation

$$\begin{aligned} &\int_\Gamma \frac{G(x,f(x),f'(x))}{\phi'(x)} \, S(k,h)\circ\phi(x)dx \\ &= \frac{hG(z_k,f(z_k),f'(z_k))}{\phi'(z_k)^2} + O(e^{-\pi d/h}) \end{aligned} \tag{6.18}$$

in which we replace $f(z_k)$ by f_k , and $f'(z_k)$ by a linear combination of the f_k which is given by combining (6.11) and (6.12). The approximate solution of (6.7) then involves the solution of a system of nonlinear algebraic equations for the f_k , which is usually not an easy problem to carry out.

Mixed conditions at an end-point require a modification of the form of f_N in (6.6). Consider, for example, changing the conditions (6.2) to

$$f(a) = \alpha \quad , \quad \beta f(b) + \gamma f'(b) = \delta \quad , \tag{6.2)'$$

where $|\beta|^2 + |\gamma|^2 > 0$. In this case (6.6) is replaced by the approximation (see Sec. 4.)

$$\begin{aligned} f(x) \cong f_N(x) &= \alpha\frac{b-x}{b-a} + A(x-a) + B(x-b)^2 \\ &+ \sum_{k=-N}^{N} c_k (x-b) S(k,h) \circ \phi(x) \end{aligned} \tag{6.6)'$$

Clearly, f_N satisfies (6.2)' at $x=a$, since the sum $\sum_{k=-N}^{N} c_k (x-b) S(k,h) \circ \phi(x)$ is zero at $x=a$ and at $x=b$. Also, f_N' exists on all of Γ , except at $x=a$. In particular, the derivative of $\sum_{k=-N}^{N} c_k (x-b) S(k,h) \circ \phi(x)$ is zero at $x=b$. Hence substituting f_N for f in the second equation in (6.2)' enables us to eliminate B in (6.6)', to get

$$\begin{aligned} f_N(x) &= \alpha\frac{b-x}{b-a} + \frac{\gamma\alpha + (b-a)\delta}{\beta(b-a)^3 + 2\gamma(b-a)} (x-a)^2 + A\omega(x) \\ &+ \sum_{k=-N}^{N} c_k (x-b) S(k,h) \circ \phi(x) \end{aligned} \tag{6.6)''$$

where

$$\omega(x) = (x-a)\left[1 - \frac{\{\beta(b-a)+\gamma\}(x-a)}{\beta(b-a)^2 + 2\gamma(b-a)}\right] \tag{6.6)'''$$

The expression (6.6)" for f_N involves $2N+2$ unknowns: $c_{-N}, c_{-N+1}, \ldots, c_N$ and A . We can thus carry out the approximate solution of (6.1)-(6.2)' by solving the system of $2N+2$ equations

$$\int_\Gamma (\mathcal{L}f_N)(x)(x-b)S(k,h)\circ\phi(x)\,\frac{dx}{\phi'(x)}\ ,$$

$$\cong h(\mathcal{L}f_N)(z_k)/\phi_k'^2 = 0\ ,\ k=-N,-N+1,\ldots,N$$

(6.7)'

$$\int_\Gamma (\mathcal{L}f_N)(x)\,\frac{\omega(x)}{\phi'(x)}\,dx$$

$$\cong h\sum_{k=-N}^{N}\frac{(\mathcal{L}f_N)(z_k)\omega(z_k)}{\phi_k'^2} = 0$$

It is tempting to deal with (6.2)' more simply by means of the approximation

$$f_N(x) = \alpha\frac{b-x}{b-a} + \mu\frac{x-a}{b-a} + \sum_{k=-N}^{N} c_k S(k,h)\circ\phi(x) \tag{6.17}$$

where μ and the c_k are unknown. Then, since the z_k "bunch up" near a and b, it is tempting to use the approximation

$$f'(b) \cong \frac{\mu-\alpha}{b-a} - \frac{c_N}{b-z_N} \tag{6.18}$$

and to take care of the second condition in (6.2)' via this approximation. However, this does not work. While (6.17) may be used as an accurate approximation of f on Γ, the c_k are close to zero for k large and positive (and also for k large and negative). Hence although the error $|f(z_N)-f_N(z_N)|$ is small,

$$\left|f'(z_N) - \frac{f_N(b)-f_N(z_N)}{b-z_N}\right| \tag{6.19}$$

may be very large (see Sec. 8.2).

The coefficients in (6.9)-(6.14), for various contour Γ corresponding to the mappings in Ex. 4.1-4.5 and the identity map of Sec. 3 are given in Table 6.1. The entries are given both as functions of x, into which we may substitute $x=z_k$ and also as functions of w, into which we may substitute $w=kh$.

TABLE 6.1. THE COEFFICIENTS IN EQS. (6-9)-(6.13)

	Γ	ϕ	$\frac{1}{\phi'^2}$	$\frac{\phi''}{\phi'^2}$
(a)	$[0,1]$	$\log \frac{x}{1-x}$ $\Longleftrightarrow \psi(w) = \frac{1}{2}+\frac{1}{2}\tanh\frac{1}{2}w$	$x^2(1-x)^2 \Longleftrightarrow \frac{e^{2w}}{(1+e^w)^4}$	2
(b)	$[-1,1]$	$\log \frac{1+x}{1-x}$ $\Longleftrightarrow \psi(w) = \tanh\frac{1}{2}w$	$\frac{1}{4}(1-x^2) \Longleftrightarrow \frac{4e^{2w}}{(1+e^w)^4}$	1
(c)	$[0,\infty]$	$\log x$ $\Longleftrightarrow \psi(w) = e^w$	$x^2 \Longleftrightarrow e^{2w}$	-1
(d)	$[0,\infty]$	$\log \sinh x$ $\Longleftrightarrow \psi(w) = \log[e^w + \sqrt{1+e^{2w}}]$	$\tanh^2 x \Longleftrightarrow \frac{e^{2w}}{1+e^{2w}}$	$-\operatorname{sech}^2 x$ $\Longleftrightarrow -\frac{1}{1+e^{2w}}$
(e)	$[-\infty,\infty]$	x	1	0

Let us illustrate the above method on the approximate solution of some "model" boundary value problems. The computations of these problems were carried out by Burke [6].

EXAMPLE 6.1:

(6.20) $$\varepsilon^2 f'' - f + 1 = 0 \ , \ 0 < x < 1 \ ; \ f(0) = f(1) = 0 \ .$$

This problem has the solution

(6.21) $$f(x) = 1 - \frac{e^{-(1-x)/\varepsilon} + e^{-x/\varepsilon}}{1+e^{-1/\varepsilon}}$$

The system (6.15) for this equation becomes

(6.22) $$\left[\frac{2}{h} I^{(2)} - 2I^{(1)} - \frac{h}{\varepsilon^2} A(x^2(1-x)^2)\right] \underset{\sim}{f} = -\frac{h}{\varepsilon^2} A(x^2(1-x)^2) \underset{\sim}{1}$$

Solving this system of equations yields the approximation

(6.23) $$f_N(x) = \sum_{k=-N}^{N} f_k S(k,h) \circ \log\left(\frac{x}{1-x}\right) \ ;$$

Taking $h = .75/N^{1/2}$, $N=16$, we get a solution which is accurate to 5 places if $\varepsilon=1/5$ and 3 places if $\varepsilon=1/10$. Similar accuracy could have been obtained if instead of solving (6.20) we had solved

(6.20)' $$\varepsilon^2 f'' - f + x^{-1}(1-x)^{-1} = 0 \ , \ f(0) = f(1) = 0 \ .$$

EXAMPLE 6.2:

(6.24) $$f'' = f - f^3/x^2 \ , \ 0 < x < \infty \ ; \ f(0) = f(\infty) = 0$$

This problem is the radially symmetric form of the three dimensional nonlinear

Klein-Gordon equation. Its solutions f satisfy $\phi' f \in B(\mathcal{D})$, where $\mathcal{D}$ is the region defined in (4.13) and where $\phi(x) = \log \sinh x$. Moreover, f is bounded on $[0,\infty]$, $f(x) = O(x)$ as $x \to 0$, $f(x) = O(e^{-x})$ as $x \to \infty$. Hence we may expect the approximation

$$f_N(x) = \sum_{k=-N}^{N} f_k \, S(k,h)\circ(\log \sinh x) \tag{6.25}$$

to be accurate. Substituting (6.25) into the differential equations and using (6.10), (6.13), (6.14) and the entries (d) of Table 6.1 we get the nonlinear system of equations

$$\begin{aligned}&[\tfrac{1}{h} I^{(2)} + A(\tfrac{1}{1+e^{2w}}) I^{(1)} - hA(\tfrac{e^{2w}}{1+e^{2w}})]\, \underset{\sim}{f} \\ &= -\, hA(\tfrac{e^{2w}}{1+e^{2w}}) A(\tfrac{1}{\{\log[e^{w}+\sqrt{1+e^{2w}}]\}^2}) \underset{\sim}{f}^3\end{aligned} \tag{6.26}$$

where $\underset{\sim}{f}^3 = (f_{-N}^3 \, f_{-N+1}^3, \ldots, f_N^3)$. Taking $h = .75/N^{1/2}$, $N=16$ and solving this system by Newton's method we are able to get the approximation of the unique positive solution of the problem (6.24) which is accurate to 5 dec. on $[0,\infty]$. The problem (6.24) has other solutions, and the system (6.26) has other solutions which approximatee these.

EXAMPLE 6.3:

$$u_{xx} + u_{yy} = -1 \, , \; (x,y) \in S = (0,1)\times(0,1) \tag{6.27}$$

$$u = 0 \text{ on } \partial S \tag{6.28}$$

$\phi(x) = \log[x/(1-x)]$ and $h = .75/N^{1/2}$. Substituting the approximation

(6.29) $$u_N(x,y) = \sum_{i=-N}^{N} \sum_{j=-N}^{N} u_{ij} S(i,h)\circ\phi(x) S(j,h)\circ\phi(y)$$

into (6.27), multiplying by $[\phi'(x)]^{-1}S(k,h)\circ\phi(x)[\phi'(y)]^{-1}S(\ell,h)\circ\phi(y)$, integrating over S and using (6.9) and (6.13) yields the system

(6.30) $$BU + UB^T = W$$

where

(6.31)
$$B = I^{(2)} - 2hI^{(1)}$$
$$U = [u_{ij}] \;,\; i,j=-N,-N+1,\ldots,N$$
$$W = -h^2 A\left(\frac{e^{2w}}{(1+e^{2w})^4}\right) E \; A\left(\frac{e^{2w}}{(1+e^{2w})^4}\right)$$
with $E = [e_{ij}] = [1] \;,\; i,j=-N,-N+1,\ldots,N$.

Setting*

(6.32) $$B = T^{-1}\Lambda T \;,\; \Lambda = \begin{pmatrix} \lambda_{-N} & & & \\ & \lambda_{-N+1} & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix}$$

we get

(6.33) $$\Lambda Y + Y\Lambda = W'$$

where

(6.34)
$$Y = [y_{ij}] = TUT^{-1}$$
$$W' = [w'_{ij}] = TWT^{-1}$$

*From the symmetry of the problem, $UB^T = BU$ and we could therefore have solved (6.30) more directly, via the formula $U = \frac{1}{2}B^{-1}W$. However the above procedure is more general.

The solution of (6.33) is

(6.35) $$Y = \frac{w'_{ij}}{\lambda_i+\lambda_j}$$

which yields

(6.36) $$U = T^{-1}Y(T^{-1})^T$$

Substituting this result into (6.29) we get an approximate solution u_N which is accurate to 5 dec. on S. Similar accuracy obtains if the -1 in (6.27) is replaced by $[x(1-x)y(1-y)]^{-1}$.

EXAMPLE 6.4:

(6.37) $$u_{xx} = u_t \quad , \ (x,t) \in (0,1)\times(0,\infty)$$

(6.38) $$u(x,0^+) = \sin(\pi x) \ .$$

The exact solution of this problem is

(6.39) $$u(x,t) = e^{-\pi^2 t}\sin \pi x$$

The Galerkin approximation

(6.40) $$u_N(x,t) = e^{-4t}\sin\pi x + \sum_{i=-N}^{N}\sum_{j=-N}^{N} u_{ij}S(i,h)\circ\phi(x)S(j,h^*)\circ\phi^*(t)$$

satisfies the boundary conditions (6.38), where * $\phi(x)=\log[x/(1-x)]$, $\phi^*(t)=\log t$.

---

*Due to the $O(e^{-at})$ rate of decrease of $u(x,t)$ as a function of t as $t \to \infty$, we could have achieved greater accuracy by taking $\phi^*(t) = \log \sinh t$, as in Ex. 6.2.

Substituting (6.40) into (6.37), we arrive at the system of equations

$$\text{(6.41)} \qquad BU + UC = V$$

where B and U are the same as in (6.30),

$$C = -\frac{h^2}{h^*} I^{(1)} A(e^{w^*})$$

(6.42)

$$V = h^2 h^* A\left(\frac{e^{2w}}{(1+e^w)^4}\right)[e^{-4t_\ell}\sin(\pi z_k)]A(e^{2w^*})$$

in which w is evaluated at kh, w^* at ℓh^* , $t_\ell = \ell^{\ell h^*}$, $z_k = \frac{1}{2}+\frac{1}{2}\tanh(jh/2)$, and where $h = .75/N^{1/2}$, $h^* = .5/N^{1/2}$, and $N=16$. The equation (6.41) is solved for U be diagonalization of B and C , and then proceeding similarly as in Ex. 6.3. The resulting approximate solution is accurate to 4 dec. on S .

7. APPROXIMATE SOLUTION OF INTEGRAL EQUATIONS.

The solution of linear integral equations, like the solution of linear differential equations, is analytic in each variable wherever the coefficients of the equation are analytic. Thus we can determine a priori, the regions on which the solution of a problem is analytic. This is usually the case for most nonlinear integral equations arising in applications. It is often more difficult to determine the exact nature of a singularity and it is in these instances that the methods of Sec. 3 and 4 are particularly powerful. In this section we illustrate the application of some of the approximations of Sec. 3 and 4 on the solution of Volterra and Fredholm integral equations.

Basic to the method of approximation is the Galerkin scheme (see [18] for a summary of this scheme for the solution of Fredholm integral equations; the case for more general linear and nonlinear equations is discussed in [44]). For our purposes, the function $S(k,h)\circ\phi(x)$ play the most important role in this scheme, and for the most important kernels arising in applications the explicit approximations of Secs. 3 and 4 enable us frequently to replace an integral equation by a system of algebraic equations without performing any numerical integration. Such procedures have been effectively carried out on the numerical solution of one and two-dimensional singular integral equations in [32, 36].

Let us consider the case of a one-dimensional problem, such as

$$f(x) = (\mathcal{K}f)(x)+g(x) \ , \ x \in \Gamma \tag{7.1}$$

where $\mathcal{K}f$ takes on one of the forms

$$(\mathcal{K}f)(x) = \int_a^x \mathcal{K}(x,t)f(t)dt \tag{7.2}$$

or

$$(7.3) \qquad (\mathcal{K}f)(x) = \int_\Gamma \mathcal{K}(x,t)f(t)dt$$

We assume that $\mathcal{D}$ is a bounded domain, that g is analytic in $\mathcal{D}$, that $g \in \mathrm{Lip}_\alpha(\bar{\mathcal{D}})$, $0 < \alpha < 1$, and that for any such g, $(\mathcal{K}g)$ has the same properties as g. We may then expect (7.1) to have a solution f with these properties. Assuming this to be the case, we approximate f and g on Γ by

$$(7.4) \qquad \begin{aligned} f(x) &\cong f_N(x) = \sum_{k=-N+1}^{N+1} c_k \psi_k(x) \\ g(x) &\cong g_N(x) = \sum_{k=-N-1}^{N+1} d_k \psi_k(x) \end{aligned}$$

where

$$(7.5) \qquad \begin{aligned} \psi_{-N-1}(x) &= \frac{b-x}{b-a} \\ \psi_k(x) &= S(k,h)\circ\phi(x) \quad , \ k=-N,-N+1,\ldots,N \\ \psi_{N+1}(x) &= \frac{x-a}{b-a} \quad . \end{aligned}$$

We then set

$$(7.6) \qquad \mu_j(x) = (\mathcal{K}\psi_j)(x) \quad , \ j=-N-1,-N,\ldots,N+1$$

and we approximate μ_j on Γ by (7.4), namely

$$(7.7) \qquad \mu_j(x) \cong \mu_{jN}(x) = \sum_{k=-N-1}^{N+1} e_{jk} \psi_k(x) \ .$$

Upon substituting these approximations into (7.1), we arrive at the system of equations

$$c_k - \sum_{j=-N-1}^{N+1} e_{jk} c_j = d_k \ , \ k=-N-1,-N,\ldots,N+1 \tag{7.8}$$

for determining the c_k .

The most difficult part of the procedure is the accurate approximation of $\mu_j(x)$ in Eq. (7.6). To this end, the approximations in Secs. 3 and 4 are frequently helpful, especially for the case of important singular integral equations arising in applications. Assuming that by taking $h = [\pi d/(\alpha N)]^{1/2}$ we can approximate g in (7.4) and μ_j in (7.6) to within an error of $O(N^{1/2}\exp[-(\pi d\alpha N)^{1/2}])$, the resulting approximation of f by f_N is accurate to within an error of $O(N^{3/2}\exp[-(\pi d\alpha N)^{1/2}])$.

Let us illustrate the solution of some integral equations via examples.

<u>EXAMPLE 7.1:</u> <u>A Volterra Integral Equation.</u> Let us consider the approximate solution of the Volterra integral equation

$$f(x) = \int_0^x [k(t)f(t)+g(t)]dt + r(x) \ , \ x \in [0,1] \ , \tag{7.9}$$

by use of the formula (4.58). Let $\mathcal{D}$ be defined by (4.9), and let k, g and $r \in B(\mathcal{D})$. We assume furthermore that $r \in \mathrm{Lip}_\alpha[0,1]$, $\alpha > 0$, and that (7.9) has a solution $f \in B(\mathcal{D}) \cap \mathrm{Lip}_\alpha[0,1]$. The method we shall describe for obtaining an approximate solution of f in (7.9) may also be applied to get an approximate solution of the initial value problem

$$\frac{d\underset{\sim}{y}}{dx} = A(x)\underset{\sim}{y} + \underset{\sim}{b}(x) \ , \ x \in (0,1) \ , \ \underset{\sim}{y}(0) = \underset{\sim}{y}_0 \tag{7.10}$$

in which $\underset{\sim}{b}$, $\underset{\sim}{y}$ and $\underset{\sim}{y}_0$ are vectors and A is a matrix.

Let us approximate f and r on [0,1] by

$$f(x) \cong f_N(x) = \sum_{j=-N-1}^{N+1} c_j \, \psi_j(x)$$

$$r(x) \cong r_N(x) = \sum_{j=-N-1}^{N+1} \rho_j \, \psi_j(x)$$

$$\psi_{-N-1}(x) = 1-x \ , \ \psi_{N+1}(x) = x$$

(7.11)
$$\psi_j(x) = S(j,h) \circ \log \frac{x}{1-x} \ , \ j=-N,-N+1,\ldots,N$$

$$\rho_{-N-1} = r(0) \ , \ \rho_{N+1} = r(1)$$

$$\rho_j = -(1-z_j)\rho_{-N-1} - z_j \rho + r(z_j)$$

$$z_j = \tfrac{1}{2}+\tfrac{1}{2} \tanh\,(jh/2) \ .$$

Substituting these expressions into (7.9), using (4.58) and making the approximation $\int_0^1 g(x)dx \cong h \sum_{j=-N}^{N} z_j(1-z_j)g(z_j)$, we get the system of equations

$$c_{-N-1} = \rho_{-N-1}$$

$$c_k = h \sum_{j=-N}^{N} z_j(1-z_j)\Big[k_j\{c_j+(1-z_j)c_{-N-1}+z_j c_{N+1}\}+g_j\Big] \cdot$$

(7.12)
$$\cdot \left| \sigma_{k-j} - h \sum_{\ell=-N}^{N} \sigma_{k-\ell} \right| + \rho_k$$

$$c_{N+1} = h \sum_{j=-N}^{N} z_j(1-z_j)\Big[k_j\{c_j+(1-z_j)c_{-N-1}+z_j c_{N+1}\}+g_j\Big]+\rho_{N+1}$$

Taking $h = [\pi d/(\alpha N)]^{1/2}$ and solving this system for the c_k we get the approximation f_N in (7.11), which differs from f by less than $C_1 N^2 e^{-(\pi d \alpha N)^{1/2}}$, where C_1 depends only on k,g,r and α .

EXAMPLE 7.2:

Consider the solution of the problem

$$(7.13) \qquad f(x) = \int_0^\infty k(x-t)f(t)dt+g(x) \ , \ x > 0 \ ,$$

where $k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\int_0^\infty |g(t)|^2 \, dt < \infty \int_0^\infty |g(t)|dt < \infty$. Let us assume that the transforms

$$(7.14) \qquad K(x) = \int_{\mathbb{R}} e^{ixt}k(t)dt \ , \ G_+(x) = \int_0^\infty e^{ixt}g(t)dt$$

can be explicitly expressed. Let $K(x) \neq 1$, and let $\int_{\mathbb{R}} d \log[1-K(x)] = 0$. Then the problem (7.13) has a unique solution f on $(0,\infty)$. For given F , let PF be defined as in (3.37). Then the Fourier transform F_+ of the solution to (7.13) may be expressed via the formula

$$(7.15) \qquad F_+ = \exp[P\Phi][G_+ + P\{G_+[\exp\{(1-P)\Phi\}-1]\}]$$

where $\Phi = -\log[1-K]$. The formula (3.48) may now be used to approximate $P\Phi$, $(1-P)\Phi$, and P of the remaining function in (7.15). Using (3.17), $f(t)$ may therefore be approximated by a truncated Fourier series on $(0, \pi/h)$. The details are carried out in [46,47]. The convergence of this approximation procedure is proved in [43,46].

EXAMPLE 7.3:

The approximations of this paper were effectively used in [36] for obtaining approximate solutions of integral equations of the form

$$f(P) = \lambda \iint_S \frac{F(P,Q)}{|P-Q|} f(Q)dA_Q + G(P), P \in S \ , \tag{7.16}$$

where S is a surface in $\mathbb{R}^3$, forming the boundary of a bounded region V . For example, the solution of the Neumann problem over a volume V with surface S

$$\Delta u = 0 \quad \text{in} \quad V \ , \ \frac{\partial u}{\partial n} = G \quad \text{on} \quad S \tag{7.17}$$

can be represented as a single layer potential

$$u(P) = \iint_S \frac{\mu(Q)}{|P-Q|} dA_Q \ , \ P \in V \ . \tag{7.18}$$

The unknown density function μ then satisfies the integral equation

$$\mu(P) + \frac{1}{2\pi} \iint_S \frac{\partial}{\partial n_P} \left(\frac{1}{|P-Q|}\right) \mu(Q) dA_Q = \frac{g(P)}{2\pi} \ , \ P \in S \ . \tag{7.19}$$

This equation has eigenvalue 1, and from the Fredholm alternative, we must have $\iint_S g(Q)dA_Q = 0$, in order for a solution to exist.

In problems of the type (7.16) or (7.19) arising in applications S is a Liapunov surface, and we generally expect to be able to subdivide S into a relatively small number of patches, $S = \cup_{\ell=1}^{L} S_\ell$, so that:

(i) For each $\ell=1,2,\ldots,L$, S_L can be parametrized over $\mathcal{S} = [-1,1]\times[-1,1]$, call the parametrization map

$$T_\ell : \mathcal{S} \to S_\ell \ . \tag{7.20}$$

Let T_ℓ take the form

$$(7.21) \qquad T_\ell(x,y) = (t_\ell^{(1)}(x,y)\ ,\ t_\ell^{(2)}(x,y)\ ,\ t_\ell^{(3)}(x,y))$$

where $(x,y) \in S$. We assume moreover that for each fixed $y(\text{rsp.}x) \in [-1,1]$, each $t_\ell^{(j)}(\cdot,y)(\text{rsp.}t_\ell^{(j)}(x,\circ)) \in B(\mathcal{D}) \cap \mathrm{Lip}_\alpha[-1,1]$, where $\alpha > 0$ and where $\mathcal{D}$ is defined as in (4.9);

(ii) If $P \in S_\ell$, then $G(T_\ell)$, satisfies the above conditions that $T_\ell^{(j)}$ satisfies.

These assumptions enable us to reduce the equation (7.16) over S to a system of integral equations over S. The approximation procedure which is then applied is most simply described by considering the case $L = 1$. In this case the resulting equation over S takes the form

$$(7.22) \qquad \mu(x,y) + \int_{-1}^{1}\int_{-1}^{1} \frac{F(x,y;\xi_1\eta)}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\,\mu(\xi,\eta) = G(x,y)$$

where $F(\cdot,y;\xi,\eta)$ and $G(\cdot,y)$ (rsp. $F(x,\cdot;\xi,\eta)$ and $G(x,\cdot)$) belong to $B(\mathcal{D}) \cap \mathrm{Lip}_\alpha[-1,1]$ as functions of x (rsp. y) with the remaining variables fixed in $[-1,1]$.

If f is as arbitrary continuous function defined on $[-1,1]$, let us set

$$(7.23) \qquad \mathcal{L}_N(f(x)) = \sum_{k=-N}^{N} f(z_k)S(k,h)\circ\phi(x)$$

where the z_k and ϕ are defined in (4.10). We then approximate μ (and similarly, G) on S by

(7.24) $$\mu(x,y) \cong \mu_N(x,y) = \sum_{j,k=-N-1}^{N+1} \mu_{jk}\, \psi_{jk}(x,y)$$

where

$$\psi_{jk}(x,y) = S(j,h)\circ\phi(x)\ \ S(k,h)\circ\phi(y)\ ,$$

$$j,k = -N,-N+1,\ldots,N$$

$$\psi_{-N-1,k}(x,y) = (\tfrac{1-x}{2})S(k,h)\circ\phi(y)$$

$$\psi_{N+1,k}(x,y) = (\tfrac{1+x}{2})S(k,h)\circ\phi(y)$$

$$\psi_{j,-N-1}(x,y) = (\tfrac{1-y}{2})S(j,h)\circ\phi(x)$$

(7.25) $$\psi_{j,N+1}(x,y) = (\tfrac{1+y}{2})S(j,h)\circ\phi(x)$$

$$\psi_{-N-1,-N-1}(x,y) = (\tfrac{1-x}{2})(\tfrac{1-y}{2})$$

$$\psi_{-N-1,N+1}(x,y) = (\tfrac{1-x}{2})(\tfrac{1+y}{2})$$

$$\psi_{N+1,-N-1}(x,y) = (\tfrac{1+x}{2})(\tfrac{1-y}{2})$$

$$\psi_{N+1,-N-1}(x,y) = (\tfrac{1+x}{2})(\tfrac{1+y}{2})$$

We are thus led to a system of $(2N+3)^2$ equation in $(2N+3)^2$ unknown. By taking $h = [\pi d/(\alpha N)]^{1/2}$ the solution of this system yields the approximation (7.24) which approximates μ on S to within an error of $O(Ne^{-(\pi d\alpha N)^{\frac{1}{2}}})$.

For example, if V is the unit ball and $g(x,y,z) = \alpha(1-3z^2)$ on S, where α = const., then (7.17) has the solution

$$u(x,y,z) = x^2+y^2-2z^2+k \tag{7.26}$$

in V, where k is an arbitrary constant. This problem was solved in [36] by use of the approximation (7.24), and the resulting matrix equation was then solved by use of Singular Value Decomposition [15]; by taking $N=2$ and using the symmetry of the problem made it possible to reduce the solution to that of a singular system of equations of order 13, yielding 3 places of accuracy.

EXAMPLE 7.4: Other Examples.

(a) In [8] the Helmholz problem $\Delta u = k^2 u$ subject to Dirichlet boundary conditions was solved on the exterior of a bounded region W in the plane, via an integral equation method, using the methods in Sec. 4. It was assumed that the boundary L of W consists of a finite number of analytic arcs L_j, with the property that the mappings ϕ_j as well as the function $g(\phi_j)$ are in $B(\mathcal{D}) \cap \mathrm{Lip}_\alpha[-1,1]$, where

$$\phi_j : [-1,1] \to L_j \quad , \tag{7.27}$$

where $\mathcal{D}$ is defined as in (4.9), and where g denotes the boundary value of u on L.

(b) In [19] the Hilbert problem

$$F_+(t) = G(t)F_-(t) + H(t) \quad , \quad t \in L \tag{7.28}$$

was solved in the complex plane via methods in Sec. 4. It was assumed in [19] that L consists of a finite number of non-overlapping closed contours in the complex plane, and that these are made up of a finite number of analytic arcs L_j which can be defined in the same fashion as the L_j in (a) above, and such that each of the functions $G(\phi_j)$ and $H(\phi_j)$ are in $B(\mathcal{D}) \cap \mathrm{Lip}_\alpha[-1,1]$, where $\mathcal{D}$ is defined in (4.9).

(c) In [32], the problem of determining the three-dimensional electric field scattered by an axially symmetric body V in a plane wave was solved via the solution of an integral equation over the surface S of B via the methods of Sec. 4 of this paper. It was assumed in [32] that the surface S is described by

$$S = \{(x,y,z) : x^2+y^2 = f(z)\}$$

where $f \in B(\mathcal{D})$, $\mathcal{D}$ as in (4.9), and such that on $(-1,1)$

$$0 < C_1(1-z^2)^{\alpha_1} \leq f(z) \leq C_2(1-z^2)^{\alpha_2} .$$

where α_1 and α_2 are in $(0,1)$.

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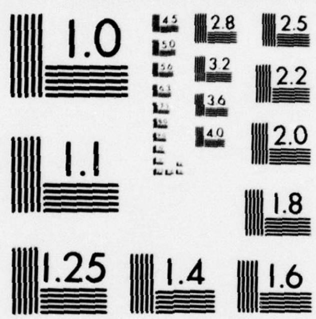

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8. COMPUTER IMPLEMENTATION AND PITFALLS.

8.1 Computer Algorithms.

Some of the formulas of the previous section have already been implemented via computer programs. Included among these is an automatic integration program over an arbitrary interval (a,b) using the formulas in Sec. 4.2 [35], programs for evaluating each of the transforms (5.1)-(5.4) [22], programs for the approximate solution of each of the problems of the examples in Sec. 6, and programs for the approximate solution of the problems in Examples 2, 3 and 4 in Sec. 7, as well as in programs for computing the solution to a Hilbert problem.

Let us briefly describe the implementation of the quadrature program [35].

Let $f \in B(\mathcal{D})$, and consider the approximation

$$\int_\Gamma f(x)dx \cong T_h(f) \equiv h \sum_{k=-\infty}^{\infty} \frac{f(z_k(h))}{\phi'(z_k(h))} \quad . \tag{8.1}$$

Let us also set

$$M_h(f) = h \sum_{k=-\infty}^{\infty} \frac{f(z_{2k-1}(h/2))}{\phi'(z_{2k-1}(h/2))} \quad , \tag{8.2}$$

so that

$$T_{(h/2)}(f) = \frac{1}{2}\,[T_h(f)+M_h(f)] \tag{8.3}$$

The error bound in (4.31) shows that when h is replaced by $\frac{1}{2}h$, the correct number of significant figures in the approximation (8.1) double.

Assume then, that we start with $h=1$ (say) and then compute $T_h(f)$. Next, we compute $M_h(f)$, so that if the difference

$$T_h(f) - M_h(f) = \varepsilon \quad , \tag{8.4}$$

then

$$T_{(h/2)}(f) = \frac{1}{2}(T_h(f) + M_h(f)] = O(\varepsilon^2) < \varepsilon \quad . \tag{8.5}$$

In practice we cannot sum all of the terms in the infinite sums (8.1) and (8.2). The assumption that

$$\frac{f(x)}{\phi'(x)} = O(e^{-\alpha|\phi(x)|}) \quad \text{on } \Gamma , \ \alpha > 0 \quad , \tag{8.6}$$

then offers a convenient stopping criteria in approximating the infinite sums. Suppose that we stop the summations (8.1) for $k > 0$ when

$$\frac{f(z_N)}{\phi'(z_N)} \ (= O(e^{-\alpha Nh})) < \varepsilon/3 \quad . \tag{8.7}$$

Then we may expect that

$$\begin{aligned} \left| h \sum_{k=N+1}^{\infty} \frac{f(z_k(h))}{\phi'(z_k(h))} \right| &\leq O(h \sum_{k=N+1}^{\infty} e^{-\alpha kh}) \\ &= O(\frac{h\, e^{-\alpha(N+1)h}}{1-e^{-\alpha h}}) \qquad (8.8) \\ &= O(e^{-\alpha Nh}) = O(\varepsilon) \quad . \end{aligned}$$

That is, we may expect the tail of the series to be of the same order of magnitude as the last included term. In order to avoid stopping the algorithm at or near a zero of f in practice, we make the more reliable test

$$\frac{|f(z_N)|}{\phi'(z_N)} + \frac{|f(z_{N+1})|}{\phi'(z_{N+1})} + \frac{|f(z_{N+2})|}{\phi'(z_{N+2})} < \frac{\varepsilon}{3} \quad . \tag{8.9}$$

A similar test is carried out for negative k since the function $f(x)/\phi'(x)$ may converge to zero at different rates, as $x \to a$ or as $x \to b$ along Γ.

These ideas form the basis for the automatic integration algorithm in [35].

8.2. Pitfalls in Computation.

The accuracy of the formulas of this paper, in spite of singularities at an end-point of an interval is based on our being able to accurately compute the function values at the points z_k . Consider, for example, the evaluation of

$$I = \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}}\, dx \tag{8.10}$$

via the formula (4.34), in 16 significant figure floating point arithmetic [51]. The points $z_k(h) = [e^{kh}-1]/[e^{kh}+1]$ cluster about $+1$ (rsp. -1) for k large and positive (rsp. negative), and the formula (4.34) may fail, due to roundoff error, resulting in the inaccurate evaluation of $[1-z_k^2(h)]^{-\frac{1}{2}}$. For example, if we take $h = \log 2$, $k=54$, the computed value of $z_k(h)$ is $.99999\ 99999\ 99999\ 0$, so that the computed value of $(1-z_k^2)^{-\frac{1}{2}}$ is $.707\cdots\times 10^8$. The actual value of $(1-z_k^2)^{-\frac{1}{2}}$ computed by means of the formula $(1-z_k^2)^{-\frac{1}{2}} = (1+e^{kh})/(2e^{kh/2})$ is $.671\cdots\times 10^8$. Hence due to roundoff, the term

$$h\left\{\frac{2e^{kh}}{(1+e^{kh})^2}[1-z_k^2]^{-\frac{1}{2}} + \frac{2e^{-kh}}{(1+e^{-kh})^2}[1-z_k^2]^{-\frac{1}{2}}\right\} \tag{8.11}$$

contributes as error of $.554\cdots\times 10^{-10}$ to the numerical approximation.

$$\log h2 \sum_{k=-54}^{54} \frac{2^{k+1}}{(1+2^k)^2}\left[1-\left(\frac{2^k-1}{2^k+1}\right)^2\right]^{-\frac{1}{2}} \tag{8.12}$$

of I . That is, it is possible to achieve no more than 10 significant figures of accuracy. If we had carried the summation from -58 to 58 instead of from -54 to 54 the situation would have been considerably worse. In that case $z_{58} = [2^{58}-1]/[2^{58}+1]$ is computed to be 1.00000 00000 00000 so that an error message results, since the computer cannot evaluate $(1.0-1.0)^{-\frac{1}{2}}$.

We emphasize that the above difficulty can be easily remedied, simply by computing the terms $[1-z_k^2]^{-\frac{1}{2}}$ by means of the expression $[1+e^{kh}]/(2e^{kh/2})$.

Similarly, the formula (see Eq. (4.14))

$$z_k = \log[e^{kh} + \sqrt{1+e^{2kh}}\,] \tag{8.13}$$

is not an accurate formula for computing z_k for the approximations (4.29), and (4.38) when $e^{kh} \leqq .01$, in that case the formula

$$z_k = e^{kh} - \frac{1}{6}e^{3kh} + \frac{3}{40}e^{5kh} - \frac{5}{112}e^{7kh} + \ldots \tag{8.14}$$

is preferable.

Accurate computation of the coefficients is equally important for the case of the formulas in Sec. 4.3, used for the approximation of derivatives over finite and semi-infinite intervals. For example, a small error in the computation of $f(z_N)$ or $(1-z_N^2)$ can cause a large error in the approximation of a derivative in the expression

$$f(x) \cong \sum_{k=-N}^{N} \frac{f(z_k)}{(1-z_k^2)^m} (1-x^2)^m S(k,h)\circ\log\frac{1+x}{1-x} \tag{8.15}$$

used to approximate $f, f', \ldots, f^{(m)}$ on $[-1,1]$.

There is one additional pitfall which we have encountered. For example, let $\mathcal{D}$ be defined as in (4.9), let $f \in B(\mathcal{D})$, and let $|f(x)| \leqq C(1-x^2)$ on $(-1,1)$, where $C > 0$. Then the approximation

$$f(x) \cong \sum_{k=-N}^{N} f(z_k)S(k,h)\circ\log\left(\frac{1+x}{1-x}\right) \tag{8.16}$$

in which $h = (\pi d/N)^{1/2}$, $z_k = (e^{kh}-1)/(e^{kh}+1)$ is accurate for moderate values of N if the numbers $f(z_k)$ are computed accurately, and moreover, the approximation

$$f'(1) \cong \frac{-f(z_N)}{1-z_N} \tag{8.17}$$

is then accurate. Suppose, for example, that f is of the order of 1 , and that the approximation (8.16) is within 10^{-5} of f for all $x \in [-1,1]$. An error of 10^{-6} in the computed values of z_k would not change this accuracy. However, since $f(x) \to 0$ as $x \to \pm 1$, we may have $f(z_N) = \frac{1}{2} \times 10^{-6}$, and an error of 10^{-6} in $f(z_N)$ will produce a very large error in the approximation (8.17). We must therefore warn against using the computed $f(z_k)$---especially those obtained as an approximate solution to a problem via the use of $S(k,h)\circ\phi(x)$ as basis functions--to approximate the derivatives of f at or near an end-point of an interval.

9. OPTIMALITY OF THE APPROXIMATIONS.

The results of this section show that the $O(e^{-cn^{\frac{1}{2}}})$ rate of convergence of the methods of this paper cannot be improved. While the functions $S(k,h)\circ\phi$ form a basis giving this rate of convergence, they are not the only ones; rational functions may also be used to achieve this rate of convergence [48,52].

The order of the error of approximate methods based on polynomials and trigonometric functions is well known (see e.g. [2]) for many classes of functions. We briefly cite some of these, for purposes of comparing methods of approximation with or without the presence of a singularity.

Let $\mathbb{P}_n$ denote the family of polynomials of degree $\leq n$.

<u>THEOREM 9.1 [2]</u>: Let $\rho > 1$, and let ε_ρ denote the ellipse with foci at ± 1 and sum of semi-ones equal to ρ . Let f be analytic and bounded in ε_ρ . Then

$$(9.1) \qquad \inf_{p\in\mathbb{P}_n} \sup_{-1<x<1} |f(x)-p(x)| = O(\rho^{-n}) \text{ , } n\to\infty \text{ .}$$

That is, the error of approximation converges to zero at the $O(e^{-cn})$ rate. The rate of convergence (9.1) is best possible with regard to order.

The rate of convergence of the error of approximation by polynomials is considerably slower if a singularity is present at an end-point of an interval. An example of this is illustrated in the following theorem.

<u>THEOREM 9.2 [2]:</u> Let $0 < \alpha < 1$. Then there exist positive constants C_1 and C_2 such that

$$\frac{c_1}{n^\alpha} \leq \inf_{p \in \mathbb{P}_n} \sup_{-1<x<1} |(1-x^2)^\alpha - p(x)| \leq \frac{c_2}{n^\alpha} \quad . \tag{9.2}$$

The same drastic change in the rate of convergence depending on whether a singularity is present or absent occurs also for quadrature formulas constructed on the basis of the formulas being exact for polynomials of a certain degree. For example, let the numbers $x_j^{(n)}$ and $w_j^{(n)}$, $j=1,2,\ldots,n$; $n=1,2,\ldots,$ be the Legendre-Gauss nodes and weights, so that the approximation

$$\int_{-1}^{1} f(x)dx \cong \sum_{j=1}^{n} w_j^{(n)} f(x_j^{(n)}) \tag{9.3}$$

is exact wherever $f \in \mathbb{P}_{2n-1}$. Then we have

<u>THEOREM 9.3 [40]:</u> Let f satisfy the conditions in Theorem 9.1. Then

$$\int_{-1}^{1} f(x)dx - \sum_{j=1}^{n} w_j^{(n)} f(x_j^{(n)}) = O(\rho^{-2n}) \ , \ n\to\infty \quad . \tag{9.4}$$

The number ρ on the right hand side of (9.4) cannot be replaced by a smaller number.

The presence of a singularity of f at ± 1 changes this rate of convergence drastically, as illustrated in the following theorem.

<u>THEOREM 9.4 [10]:</u> Let $w_j^{(n)}$ and $x_j^{(n)}$ be defined as in (9.3). If $\alpha > 0$ and not an integer, then

$$\int_{-1}^{1} (1-x)^{\alpha-1} dx - \sum_{j=1}^{n} w_j^{(n)} (1-x_j^{(n)})^{\alpha-1} \sim \frac{c(\alpha)}{n^{2\alpha}} \ , \ n\to\infty \tag{9.5}$$

where $c(\alpha)$ depends only on α.

For the methods of this paper, the error converges to zero at the $O(e^{-cn^{1/2}})$ rate, whether or not singularities are present at an end-point of an interval. The best value of c in this $O(e^{-cn^{1/2}})$ rate depends somewhat, but only mildly, on the particular type of singularity.

We may therefore seek after methods which work well in spite of the presence of a large class of singularities, and then try to determine which, among these are the best.

Let us first choose a space of functions with singularities, for purposes of approximation on $[-1,1]$.

If $p > 1$, the space $H_p(U)$ is a convenient and well known space of functions in analytic function theory. $H_p(U)$ consists of the family of all functions f that are analytic in the unit disc U in the complex plane, and for which

$$(9.6) \qquad \|f\|_p = \lim_{r\to 1^-} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty \ .$$

Thus $H_p(U)$ contains functions which may or may not have singularities at the end-points of the interval $[-1,1]$, such as $f(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\log(1-x)$, where $\alpha,\beta < 1/p$, or $f(x) = e^{-x}$, etc. The closer p is to 1 , the larger the space $H_p(U)$, since, for example, if $1 < p' < p$, and if $f \in H_p(U)$, then $f \in H_{p'}(U)$, whereas $H_{p'}(U)$ has in it functions that are not in $H_p(U)$, such as, for example, the function $f(x) = (1-x^2)^{-1/p}$.

Next, given the space $H_p(U)$, let us consider the error of approximation

$$(9.7) \qquad I(f) - Q_n(f)$$

where $f \in H_p(U)$,

$$(9.8) \qquad I(f) = \int_{-1}^{1} f(x)dx \ , \ Q_n(f) = \sum_{j=1}^{n} w_j^{(n)} f(x_j^{(n)})$$

and where $w_j^{(n)} \in \mathbb{C}$, $x_j^{(n)} \in U$. Let us set

$$(9.9) \quad \sigma_{p,n} = \inf_{w_j^{(n)} \in \mathbb{C}, x_j^{(n)} \in U} \left\{ \sup_{f \in H_p(U), \|f\|_p = 1} \left| I(f) - Q_n(f) \right| \right\}$$

The numbers $\sigma_{p,n}$ determine the best possible rate of convergence to zero of the quadrature error. At this time, the exact values of the $\sigma_{p,n}$ and the corresponding quadrature rules $Q_n = Q_n{}^*$ for which $|I(f) - Q_n{}^*(f)| \leqq \sigma_{p,n} \|f\|_p$ for all $f \in H_p(U)$ are not known. A number of papers have been written on the estimation of upper bounds for $\sigma_{p,n}$ [4,5,17,20,23,50,52,58] and some have also been written on lower bounds [4,50,52]. The results of the following theorem give the best bounds known to date.

<u>THEOREM 9.5 [52]:</u> Let $q = p/(p-1)$. Given any $\varepsilon > 0$ there exists an integer $n(\varepsilon) \geqq 0$ such that whenever $n > n(\varepsilon)$, then

$$(9.10) \qquad \exp[-(5^{1/2}\pi + \varepsilon)n^{1/2}] \leqq \sigma_{p,n} \leqq \exp[-\{\frac{\pi}{(2q)^{\frac{1}{2}}} - \varepsilon\}n^{1/2}] \ .$$

We remark that the formulas (4.34) of the present paper converge at the rate on the extreme right hand side of (9.10), for every $p > 1$, and the formulas in [4,48] also converge at this rate. No formulas are known at this time which converge at a faster rate.

Next, for purposes of interpolation on $[-1,1]$, let $p > 1$, let $H_p{}^*(U)$ denote the family of all functions g such that $f \in H_p(U)$, where $f(z) = g(z)/(1-z^2)$, and let $H_p{}^*(U)$ be normed by $\|g\|_p^* = \|f\|_p$, where $\|f\|_p$

is defined in (4.6). Let $\{T_n\}_{n=1}^{\infty}$ be a linear interpolation scheme defined by

$$T_n(g)(x) = \sum_{j=1}^{n} g(x_j^{(n)})\phi_{n,j}(x) \tag{9.11}$$

where $x_j^{(n)} \in U$, where $\phi_{n,j}$ is analytic in U for each n and j, and such that

$$\|T_n(g)\|_p^* \leq C\|g\|_p^* \tag{9.12}$$

for all $g \in H_p^*(U)$, where C is a constant independent of n. Let us set

$$\tau_{p,n} = \inf_{T_n} \sup_{f\in H_p^*(U),\|f\|_p^*=1} \sup_{-1<x<1} |f(x)-T_n(f)(x)| . \tag{9.13}$$

<u>THEOREM 9.6 [52]:</u> Given any $\varepsilon > 0$, there exists an integer $n(\varepsilon)\geq 0$, such that whenever $n > n(\varepsilon)$ and $q=p/(p-1)$,

$$\exp[-(5^{1/2}\pi+\varepsilon)n^{1/2}] \leq \tau_{p,n} \leq \exp[-(\frac{\pi}{2q^{1/2}} -\varepsilon)n^{1/2}] \tag{9.14}$$

These upper and lower bounds are the best ones known to date. The formulas (4.19) of the present paper with $\phi(x) = \log[(1+x)/(1-x)]$ converge at the rate on the extreme right of (9.14). A rational function has also been constructed for interpolation of functions in $H_p^*(U)$ over $[-1,1]$ [52]; this also converges at the rate on the extreme right of (9.14). No formulas are known at this time which converge at a faster rate.

Theorem 9.1-9.4 show that the formulas of this paper are not as good as polynomials in the absence of singularities, but they are much better when singularities are present.

In [52] one finds definitions of other H_p spaces of functions with the property that each of the formulas of Sec. 4.2 (rsp. Sec. 4.1) enjoys the $O(\exp[-\{\frac{\pi}{(2q)^{\frac{1}{2}}} - \varepsilon\} n^{1/2}])$ (rsp. $O(\exp[-\{\frac{\pi}{2q^{\frac{1}{2}}} - \varepsilon\} n^{1/2}])$ rate of convergence in these spaces.

REFERENCES

[1] Abramowitz, A., I. B. Stegun , *Handbook of Mathematical Functions*, NBS Applied Math. Series 53 Washington, D.C. (1964).

[2] Bernstein, S. and C. de la Vallee Poussin, *L'Approximation*, Chelsea N.Y. (1970).

[3] Bleistein, N. and R. A. Handelsman, *Asymptotic Expansion of Integrals*, Holt, Rinehart and Winston, New York (1975).

[4] Bojanov, B. D., *The Optimal Method for the Integration of a Certain Class of Analytic Functions*, (English summary) Math. and Math. Education, (Proc. Second Spring Conference of the Bulgarian Math. Society, Vidin (1973)(Bulgarian)75-86.) Izdat B"lgar. Akad. Nauk, Sofia (1974).

[5] Bojanov, B. D., *Best Quadrature Formula for a Certain Class of Analytic Functions*, Zastoswania Matematyki Appl. Mat. XIV (1974) 441-447.

[6] Burke, R., *Solution of Boundary Value Problems Via the Sinc-Galerkin Method*, M.Sc. Thesis, Univ. of Utah (1977).

[7] Cauer, W., *Bemerkungen über eine Extremalanfgabe*, Math. Mech. 20 (1946) 358.

[8] Chiu, Y. H., *An Integral Equation Method of Solution of* $\Delta u+k^2u=0$, Ph.D. Thesis, Univ. of Utah (1976).

[9] Davis, P., *Interpolation and Approximation*, Blaisdell, Waltham, Mass. (1963).

[10] Donaldson, J. D. and D. Elliott, *Quadrature II: The Estimation of Remainders in Certain Quadrature Rules*, Tech. Rep. 24. Dept. of Math., Univ. of Tasmania, Hobart (1970).

[11] Gakhov, F. D., *Boundary Value Problems,* translated from Russian by I. N. Sneddon, Pergamon Press, Oxford-London-New York (1966).

[12] Gončar, A. A.: *Estimates on the Growth of Rational Functions and Some of Their Applications*, Math. U.S.S.R. Sbornik 1 (1967) 445-456.

[13] Gončar, A. A., *On the Rapidity of Rational Approximation of Continuous Functions with Characteristic Singularities*, Math. U.S.S.R. Sbornik 2 (1967) 561-568.

[14] Goodwin, E. T., *The Evaluation of Integrals of the form* $\int_{-\infty}^{\infty} a(x)e^{-x^2}dx$, Proc. Camb. Philos. Soc. 45 (1949) 241-245.

[15] Galub, G. H. and C. Reinsch, *Singular Value Decomposition and Least Squares Solutions*, Numer. Math. 44 (1970) 402-420.

[16] Hartley, R. V. L., *The Transmission of Information*, Bell System Tech. J. 7 (1928) 535-560.

[17] Haber, S., *The Error in Numerical Integration of Analytic Functions*, Quart. Appl. Math. 29 411-420 (1971).

[18] Ikebe, Y., *The Galerkin Method for Numerical Solution of Fredholm Integral Equations of the Second Kind*, SIAM Rev. 14 (1972) 465-491.

[19] Ikebe, Y., T. Y. Li and F. Stenger, *The Numerical Solution of the Hilbert Problem.* In: *Theory of Approximations with Applications* (A. G. Law, B. N. Sahney, eds.) London-New York: Academic Press 1976.

[20] Johnson, L. W. and R. D. Riess, *Minimal Quadratures for Functions of Low Order Continuity*, Math. Comp. 25 (1971) 831-835.

[21] Kearfott, B. and F. Stenger, *A Sinc Approximation for the Indefinite Integral*, in manuscript.

[22] Kresz, R., *Interpolation auf einem unendlichen Intervall*, Computing 6 (1970) 274-288.

[23] Loeb, H. L. and H. Werner, *Optimal Numerical Quadratures in* H_p *Spaces*, Math. Z. 138 (1974) 111-117.

[24] Lund, J. R., *Numerical Evaluation of Integral Transforms*, Ph.D. Thesis, Univ. of Utah (1978).

[25] Lundin, L. and F. Stenger, *Cardinal-Type Approximation of a Function and Its Derivatives*, to appear in SIAM J. Math. Anal.

[26] Martensen, E., *Auf numerischen Auswertung Unendlicher Integrale*, ZAMM 48 (1968) T83-T85.

[27] McNamee, J., F. Stenger and E. L. Whitney, *Whittaker's Cardinal Function in Retrospect*, Math. Comp. 25 (1971) 141-154.

[28] Moran, P. A. P., *Approximate Relation Between Series and Integrals*, Math. Comp. 12 (1958) 34-37.

[29] Newman, D. J., *Rational Approximation to* $|x|$, Michigan Math. J. 11 (1964) 11-14.

[30] Nyquist, H., *Certain Topics in Telegraph Transmission Theory*, Trans. Amer. Inst. Elec. Engrg. 47 (1928) 617-644.

[31] Olver, F. W. J., *Asymptotics and Special Functions*, Academic Press, New York and London (1974).

[32] Petrick, W., J. Schwing and F. Stenger, *Algorithm for the Electromagnetic Scattering from Rotationally Symmetric Bodies with an Impedance Boundary Condition*, to appear in J. Math. Anal. Appl.

[33] Rahman, Q. I. and G. Schmeisser, *Minimization of* $\int_0^1 x^4|p(x)/p(-x)|^2dx$, Private communication.

[34] Schwarz, C., *Numerical Integration of Analytic Functions*, Comput. Phys. 4 (1967) 191-201.

[35] Schwing, J. and F. Stenger, *An Automatic Quadrature Algorithm Based on the Trapezoidal Formula*, in manuscript (1975).

[36] Schwing, J., *Integral Equation Method of Solution of Potential Theory Problems in* R^3 , Ph.D. Thesis, Univ. of Utah (1976).

[37] Shannon, C. E., *A Mathematical Theory of Communication*, Bell System Tech. J. 27 (1948) 379-423, 623-656.

[38] Sobolev, S. L., *Convergence of Approximate Integration Formulas from* $L_2^{(m)}$. Soviet Math. Dokl. 6 (1965) 865-867.

[39] Squire, W., *Numerical Evaluation of Integrals Using Moran Transformations*, Aerospace Engrg. Report No. TR-14, 1969.

[40] Stenger, F., *Bounds on the Error of Gauss-Type Quadratures*, Numer. Math. 8 (1966) 150-160.

[41] Stenger, F., *The Asymptotic Approximation of Certain Integrals*, SIAM J. Math. Anal. 1 (1970) 392-404.

[42] Stenger, F., *Transform Methods of Obtaining Asymptotic Expansions of Definite Integrals*, SIAM J. Math. Anal. 3 (1972) 20-30.

[43] Stenger, F., *Approximate Solution of Wiener-Hopf Integral Equations*, J. Math. Anal. Appl. 37 (1972) 687-724.

[44] Stenger, F., *Convergence and Error of the Bubnov-Galerkin Method, Proceedings on the Conference on the Numerical Solution of Differential Equations*, Springer-Verlag Lecture Notes in Math. no. 362 (1972) 434-450.

[45] Stenger, F., *Integration Formulas via the Trapezoidal Formula*, J. Inst. Maths. Applics 12 (1973) 103-114.

[46] Stenger, F., *The Approximate Solution of Convolution-Type Integral Equations*, SIAM J. Math. Anal. 4 (1973) 103-114.

[47] Stenger, F., *An Algorithm for the Approximate Solution of Wiener-Hopf Integral Equations*, Comm. ACM 16 (1973) 708-710.

[48] Stenger, F., *An Analytic Function which is an Approximate Characteristic Function*, SIAM J. Numer. Anal. 12 (1975) 239-254.

[49] Stenger, F., *Approximations via Whittaker's Cardinal Function*, J. Approx. Theory 17 (1976) 222-240.

[50] Stenger, F., *Upper and Lower Estimates on the Rate of Convergence of Approximations in* H_p , Bull. AMS, (1977), 45-148.

[51] Stenger, F., *Remarks on Integration Formulas Based on the Trapezoidal Formula,* J. Inst. Maths. Applics, 19 (1977) 145-147.

[52] Stenger, F., *Optimal Convergence of Minimum-Norm Approximations in* H_p , Numer. Math. 29 (1978) 345-362.

[53] Stenger, F., *A Sinc-Galerkin Method of Solution of Boundary Value Problems,* to appear in Math. Comp. (1979).

[54] Takahasi, H. and M. Mori, *Quadrature Formulas Obtained by Variable Transformation,* Numer. Math. 21 (1973) 206-219.

[55] Whittaker, E. T., *On the Functions which are Represented by the Expansions of the Interpolation Theory,* Proc. Roy. Soc. Edinburgh 35 (1915) 181-194.

[56] Whittaker, J. M., *On the Cardinal Function of Interpolation Theory,* Proc. Edinburgh. Math. Soc. Ser. 1 2 (1927) 41-46.

[57] Whittaker, J. M., *Interpolatory Function Theory,* Cambridge, London (1935).

[58] Wilf, H., *Exactness Conditions in Numerical Quadrature,* Numer. Math. 6 (1964) 315-319.

[59] Wong, R., *Error Bounds for Asymptotic Expansions of Hankel Transforms,* SIAM J. Math. Anal. 7 (1976) 799-808.

[60] Zensykbaev, A. A., *On the Best Quadrature Formula on the Class* W^1L_p , Soviet Math. Dokl 17 (1976) 377-380.